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On Projection Properties of Monotone Integrable Functions

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This research formulates an $(i-1,i)$ - dimensional structure of $\mu_{|f|^p}^{(i-1,i)}$-vector measure integrable functions for $i = 1, 2, ...n$. Fixed point projection properties of a vector measure are appplied to determine the measurability of sets in the domain of integrable functions. Measurable sets of the form $\Pi_i A_{i-1}^{(i,i+1)}$ are partitioned into disjoint sets $\Pi_i A_{i-1}^i$ of finite measure.The obtained results demonstrate utility of concepts of vector measure duality, continuity from below of a measure and monotonicity of a vector measure in integrating functions.

Keywords: Projection properties; measure space; integrable functions.

1 Introduction

This paper considers a sequence of monotone functions and integrability concepts of integrable functions with respect to $\mu_{|f|^p}^{(i-1,i)}$-vector measure. The utility of concepts such as vector measure duality, continuity from below

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and monotonicity of a vector measure are applied in constructing $\mu^{(i-1,i)}_{|f|^p}$-vector measurable sets with respect to the sigma ring $\rho^{(i-1,i)}$ of subsets of an n-dimensional spcace X^n where f is an integrable function with respect to a measure $\mu^{(i-1,i)}$ defined on $\rho^{(i-1,i)}$.

This study involves partitioning of measurable sets into disjoint sets. The research further apllies projection properties of vector measure duality with values in a Hilbert space.

2 Preliminaries

Definition 1(p **-Integrable Function)** (Sanchez [9])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space where $\mu^{(i-1,i)}$ is a measure defined on a sigma ring $\rho^{(i-1,i)}$ of subsets of $X \times X$. Then for $\Pi_i A^i_{i-1} \in \rho^{(i-1,i)}$ there exists a function f defined on $X \times X$ such that $\mu^{(i-1,i)}_{|f|^p}(\Pi_i A^i_{i-1}) \in Z$ where Z is a Hilbert space and $\Pi_i A^i_{i-1}$ is the product of A_i for $i = 1, 2, ...n$. The function f defined on $X \times X$ is said to be p-integrable with respect to $\mu^{(i-1,i)}$ if

$< \mu^{(i-1,i)}_{|f|^p}(\Pi_i A^i_{i-1}), z' >< \infty$

where z' is an element in Z', the dual space of Z.

Definition 2 (Vector Measure)(Otanga [6])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $L_P(\mu^{(i-1,i)})$ is the function space of p-integrable functions with respect to $\mu^{(i-1,i)}$, $\Pi_{i=1} A^i_{i-1} \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $\mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1} A^i_{i-1}) \in Z$ where Z is a Hilbert space, then the set function $\mu^{(i-1,i)}_{|f|^p} : \rho^{(i-1,i)} \to Z$ is called a vector measure.

Definition 3 (Norm of p **-Integrable Functions)**(Sanchez [9])

The set $L_P(\mu^{(i-1,i)})$ of p-integrable functions with respect to $\mu^{(i-1,i)}$ defines an order continuous Hilbert function space whose norm is given by

$\parallel f \parallel_p = sup(< \mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1} A^i_{i-1}), z' >)^{1\backslash p}$

where $\Pi_{i=1} A^i_{i-1} \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $z' \in Z'$.

Definition 4 (k_{i+1}**-Projection Product Measure)**(Otanga [5])

Let $\mu^{(i,i+1)}_{i-1}$ represent the product measure $\mu_{i-1} \times \mu_i \times \mu_{i+1}$ defined on a sigma ring $\rho^{(i,i+1)}_{i-1}$ of subsets of an $i+1$-dimensional space for $i = 1, 2, ..n$. For a fixed positive integer k_{i+1}, the set function μ^i_{i-1} where $i = 1, 2, ..n$ is called the projection product measure and is denoted by

$proj_{k_{i+1}}(\mu^{(i,i+1)}_{i-1})$

Definition 5 ($(\mu^{i-1,i})_{|f|^p}$**-Measurable Set)**(Sanchez [9])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $\Pi_i A^i_{i-1}$ is a measurable set with respect to $\rho^{(i-1,i)}$, then

$\mu^{(i-1,i)}(\Pi_i A^i_{i-1}) = \mu_{i-1}(A_{i-1}) \times \mu_i(A_i)$ for $i = 1, 2, ...n$

If $f \in L_P(\mu^i_{i-1})$ then for a fixed positive integer k_{i+1}, the set $\Pi_i A^i_{i-1}$ is said to be $(\mu^{i-1,i})_{|f|^p}$-measurable if

$< \mu^{(i-1,i)}_{|f|^p}(\Pi_i A^i_{i-1}), z' >$ is finite for $i = 1, 2, ...n$

Definition 6 (k_{i+1}**-Projection of a Measurable Set)**(Otanga [6])

Let $\Pi_{i=1} A^{(i,i+1)}_{i-1}$ be a measurable set with respect to $\rho^{(i-1,i,i+1)}$. Then the k_{i+1}-projection $\Pi_{i=1} A^i_{i-1}$ of $\Pi_{i=1} A^{(i,i+1)}_{i-1}$ is denoted by $proj_{k_{i+1}}(\Pi_{i=1} A^{(i,i+1)}_{i-1})$ where k_{i+1} is a fixed positive integer.

Definition 7 (Monotone p-Integrable Functions)

According to the results in (Otanga [5] and Sanchez [9]), a sequence $(f_n)_{n=1}^{\infty}$ of p-integrable functions is said to be monotonically increasing if $\Pi_{i=1}A_{i-1}^i \subseteq \Pi_{i=1}B_{i-1}^i$ for $i = 1, 2, ...n$ implies that

$< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' >^{1\backslash p} \leq < \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}B_{i-1}^i), z' >^{1\backslash p}$

Similarly a sequence $(f_n)_{n=1}^{\infty}$ of p-integrable functions is said to be monotonically decreasing if

$\Pi_{i=1}A_{i-1}^i \subseteq \Pi_{i=1}B_{i-1}^i$ for $i = 1, 2, ...n$ implies that

$< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' >^{1\backslash p} \geq < \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}B_{i-1}^i), z' >^{1\backslash p}$

3 Main Results

Proposition 1

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)_{n=1}^{\infty}$ be a monotonically decreasing sequence of p-integrable functions with respect to $\mu^{(i-1,i)}$. If $f_n \downarrow 0$ for each n and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$ for all (x_{i-1}, x_i), then $< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >)^{1\backslash p}$ is monotonically decreasing to zero for $i = 1, 2, ..., n$

Proof

Let $proj_{k_{i+1}}(\Pi_{i=1}E_{n}{}_{i-1}^{(i,i+1)}) = \Pi_{i=1}E_{n\,i-1}{}^i$ such that

$\Pi_{i=1}E_{n\,i-1}{}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i)) > \epsilon)$

where $\epsilon > 0$ and $f_{n+1} \leq f_n$ for all n. It follows that

$\Pi_{i=1}E_{n\,i-1}{}^i \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$

As a consequence of $f_n(x) \downarrow 0$ and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, it follows

that $\Pi_{i=1}E_{n\,i-1}{}^i \downarrow 0$ for all n (Lech [2])

Let $\Pi_{i=1}E_{i-1}^i = ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \geq f_n(x_{i-1}, x_i))$.

If $(x_{i-1}, x_i) \in \Pi_{i=1}E_{i-1}^i$, then $(f_n \cap f_1)(x_{i-1}, x_i) = f_n(x_{i-1}, x_i)$

Therefore

$((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) \neq 0) \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$

For each set $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, we have

$\chi_{((x_{i-1},x_i):f_1(x_{i-1},x_i)\neq 0)} f_n = f_n$ for $i = 1, 2, ...n$

Applying the results on integrable functions (Sanchez [9] and okada [3]) and vector duality functions (Campo *et. al.* [1]), we obtain

$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >^{1\backslash p} = < \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1}E_{n\,i-1}{}^i)^c, z' >)^{1\backslash p}$

$+ < \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^i), z' >)^{1\backslash p}$ (*)

where $(\Pi_{i=1}E_{n\,i-1}{}^i)^c$ represents the complement of $\Pi_{i=1}E_{n\,i-1}{}^i$ in the set

$((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$

Given that $f_n(x_{i-1}, x_i) \leq \epsilon$ on $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \setminus \Pi_{i=1}E_{n\,i-1}{}^i)$, it follows that

$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1} E_{n\,i-1}{}^i)^c, z' >)^{1\backslash p}$

$\leq \epsilon < \mu^{(i-1,i)}(((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1} E_{n\,i-1}{}^i)^c), z' >$

$\leq \epsilon < \mu_{i-1}^i((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >$

Let $M = sup\ (| f_n(x_{i-1}, x_i) | \forall (x_{i-1}, x_i)$. Then

$< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} E_{n\,i-1}{}^i), z' >)^{1\backslash p} \leq M < \mu^{(i-1,i)}(\Pi_{i=1} E_{n\,i-1}{}^i), z' >$ for all n

Therefore, equation (*) becomes

$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >^{1\backslash p}$

$\leq \epsilon < \mu^{(i-1,i)}(x_{i-1}, x_i : f_1(x_{i-1}, x_i) \neq 0), z' >$

$+\ M < \mu^{(i-1,i)}(\Pi_{i=1} E_{n\,i-1}{}^i), z' >$

Since ϵ is arbitrary and $< \mu^{(i-1,i)}(\Pi_{i=1} E_{n\,i-1}{}^i), z' >\downarrow 0$ for each n, it follows that

$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i : f_1(x_{i-1}, x_i) \neq 0), z' >^{1\backslash p}\downarrow 0$ for $i = 1, 2, ...n$

Proposition 2

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f and g be positive p-integrable functions with respect to $\mu^{(i-1,i)}$. If $\Pi_{i=1} E_{i-1}^i = (x_{i-1}, x_i : g(x_{i-1}, x_i) \geq f(x_{i-1}, x_i))$, then

$\| f \|_p \leq \| g \|_p$

Proof

Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be monotonically increasing p-integrable functions such that $\chi_{\Pi_{i=1} A_{i-1}^i} g_n \uparrow \chi_{\Pi_{i=1} A_{i-1}^i} g$ and $\chi_{\Pi_{i=1} A_{i-1}^i} f_n \uparrow \chi_{\Pi_{i=1} A_{i-1}^i} f$ for each n and for every measurable set $\Pi_{i=1} A_{i-1}^i$ of finite measure.

Let $< \mu_{|g_n|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >^{1\backslash p} \leq M$ for each n and $M > 0$.

If $(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$

$= (\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : (f_n \cap g_n)(x_{i-1}, x_i) \neq 0)$, then

$(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$ is a subset of

$(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : g_n(x_{i-1}, x_i) \neq 0)$

Therefore

$< \mu_{|h_n|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >^{1\backslash p} \leq M$

If $(x_{i-1}, x_i) \in \Pi_{i=1} E_{i-1}^i$,then

$(f \cap g)(x_{i-1}, x_i) = f(x_{i-1}, x_i)$

It follows that

$\Pi_{i=1} A_{i-1}^i \cap (x \in X : h_n(x) \neq 0)$ is monotonically increasing to

$(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : (f \cap g)(x_{i-1}, x_i) \neq 0)$

$= (\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Therefore

$< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >^{1\backslash p} = \text{LUB} < \mu_{|h_n|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >^{1\backslash p}$

$\leq LUB < \mu^{(i-1,i)}_{|g_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} = < \mu^{(i-1,i)}_{|g|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p}$

Taking the supremum on both sides of the inequality, (Sanchez [9]) we obtain

$\| f \|_p \leq \| g \|_p$

Proposition 3

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)^{\infty}_{n=1}$ be a sequence of positive bounded p-integrable functions with respect to $\mu^{(i-1,i)}$ such that $f_n \uparrow f$ for each n. If $\Pi_i E^i_{i-1} = ((x_{i-1}, x_i) : f((x_{i-1}, x_i)) > \epsilon)$, then $< \mu^{(i-1,i)}(\Pi_i E^i_{i-1}), z' >$ is bounded.

Proof

Since $f_n \uparrow f$ for each n (by hypothesis), it follows that $f = LUBf_n$ and $f = (f_n)^{\infty}_{n=1}$

Let $\Pi_{i=1}E_{n\,i-1}{}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) > \epsilon)$ such that

$< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_i E_{n\,i-1}{}^i), z' >^{1\backslash p} \leq M$ for all n and $M > 0$

It follows that $\Pi_{i=1}E_{n\,i-1}{}^i \uparrow \Pi_{i=1}E^i_{1-1}$ for each n

Since $f_n(x_{i-1}, x_i) > \epsilon$ for each (x_{i-1}, x_i), it follows that

$\epsilon < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^i)), z' > \leq < \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E_{n\,i-1}{}^i), z' >^{1\backslash p} \leq M$

Let $(\Pi_{i=1}F_{n\,i-1}{}^i)^{\infty}_{n=1}$ be a sequence of mutually disjoint sets such that

$\Pi_{i=1}E^i_{i-1} = \bigcup^{\infty}_{n=1} \Pi_{i=1}F_{n\,i-1}{}^i$

On application of the results in (Rodriguez [8] and Otanga [7]) and by finiteness of a vector measure (Otanga [4] and Yaogan [10]), we obtain

$< \mu^{(i-1,i)}(\Pi_{i=1}E^i_{i-1}), z' >= \sum^{\infty}_{k=1} < \mu(\Pi_{i=1}F_{k\,i-1}{}^i), z' >$

$= LUB_n \sum^{n}_{k=1} < \mu^{(i-1,i)}(\Pi_{i=1}F_{k\,i-1}{}^i), z' >$

$= LUB_n < \mu^{(i-1,i)}(\bigcup^{n}_{k=1} \Pi_{i=1}F_{k\,i-1}{}^i), z' >$

$= LUB_n < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^i), z' > \leq M$

Proposition 4

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p-integrable function with respect to $\mu^{(i-1,i)}$ and $(\Pi_{i=1}E_{n\,i-1}{}^i)^{\infty}_{n=1}$ be a sequence of measurable sets such that

$\Pi_{i=1}E_{n\,i-1}{}^i = ((x_{i-1}, x_i) :| f(x_{i-1}, x_i) | \geq 1 \setminus n)$ for each n.

If $\Pi_{i=1}E_{n\,i-1}{}^i$ is a $\mu^{(i-1,i)}_{|f|^p}$ - null set for each n, then

$< \mu^{(i-1,i)}((x_{i-1}, x_i) : f(x) \neq 0), z' >= 0$

Proof

Consider the measurable sets $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$ and $\Pi_{i=1}E_{n\,i-1}{}^i = ((x_{i-1}, x_i) :| f(x_{i-1}, x_i) | \geq 1 \setminus n)$

such that $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = LUB_n \Pi_{i=1}E_{n\,i-1}{}^i$

It follows that

$\Pi_{i=1}E_{n\,i-1}{}^i \uparrow ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Let $G_{k_i} \bigcap G_{k_j} = \emptyset$ for $k_i \neq k_j$ where $k_i, k_j = 1, 2, ...$ and $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = \bigcup^{\infty}_{k=1} \Pi_{i=1}G_{k\,i-1}{}^i$

By the property of countable additivity of a vector measure (Otanga *et. al.* [5]), we obtain $< \mu^{(i-1,i)}((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0), z' >$

$= \sum_{k=1}^{\infty} < \mu^{(i-1,i)}(\Pi_{i=1}G_{k\,i-1}{}^{i}), z' >$

$= LUB_n \sum_{k=1}^{n} < \mu^{(i-1,i)}(\Pi_{i=1}G_{k\,i-1}{}^{i}), z' >$

$= LUB_n < \mu^{(i-1,i)}(\bigcup_{k=1}^{n} \Pi_{i=1}G_{k\,i-1}{}^{i}), z' >$

$= LUB_n < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' >$

Since $1 \setminus n \leq\mid f(x_{i-1},x_i) \mid$ on $\Pi_{i=1}E_{n\,i-1}{}^{i}$ and $\Pi_{i=1}E_{n\,i-1}{}^{i}$ is a $\mu^{(i-1,i)}_{|f|^p}$ - null set for each n (by hypothesis), then

$1 \setminus n < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' > \leq < \mu^{(i-1,i)}\mid f \mid^p(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' >^{1\setminus p} = 0,$

Therefore $< \mu^{(i-1,i)}(x_{i-1},x_i) : f(x) \neq 0), z' >= 0$

Proposition 5

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p-integrable function with respect to $\mu^{(i-1,i)}$ and $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$ be a $\mu^{(i-1,i)}_{|f|^p}$ - null set, then $f = 0$ on the complement of set $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$

Proof

Let $\Pi_{i=1}G^{i}_{i-1} = ((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$, $\Pi_{i=1}E^{i}_{i-1} = ((x_{i-1},x_i) : f(x_{i-1},x_i) > 0$ and $\Pi_{i=1}F^{i}_{i-1} = ((x_{i-1},x_i) : f(x_{i-1},x_i) < 0)$ be measurable sets with respect to $\rho^{(i-1,i)}$. Since $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$ is a $\mu^{(i-1,i)}_{|f|^p}$ - null set (by hypothesis), then $< \mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1}G^{i}_{i-1}), z' >^{1\setminus p} = 0$. Since $f(x) > 0$ for each $(x_{i-1},x_i) \in \Pi_{i=1}E^{i}_{i-1}$, then $< \mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1}E^{i}_{i-1}), z' >^{1\setminus p} = 0$ and $f(x_{i-1},x_i) < 0$ for each $(x_{i-1},x_i) \in \Pi_{i=1}F^{i}_{i-1}$ implies that $< \mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1}F^{i}_{i-1}), z' >^{1\setminus p} = 0$.

It follows that

$\Pi_{i=1}G^{i}_{i-1} = ((x_{i-1},x_i) : f(x_{i-1},x_i) > 0) \cup ((x_{i-1},x_i) : f(x_{i-1},x_i) < 0)$ is a $\mu^{(i-1,i)}_{|f|^p}$ - null set

Therefore

$f = 0$ on the complement of $\Pi_{i=1}G^{i}_{i-1} = ((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$

Corollary 1

Let $(f_n)_{n=1}^{\infty}$ be a sequence of monotonically increasing p-integrable functions such that $< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E^{i}_{i-1}), z' >^{1\setminus p}$ is bounded for each n. Let $\Pi_{i=1}E_{n\,i-1}{}^{i}$ be monotonically increasing to $\Pi_{i=1}E^{i}_{i-1}$ where $\mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^{i}) < \infty$ for all n and $\Pi_{i=1}E^{i}_{i-1} = \cap_{n=1}^{\infty}\Pi_{i=1}E_{n\,i-1}{}^{i}$. If $\Pi_{i=1}E_{n\,i-1}{}^{i} = ((x_{i-1},x_i) : f_n(x_{i-1},x_i) \geq M)$ for $M > 0$, then $\Pi_{i=1}E^{i}_{i-1}$ is a $< \mu^{(i-1,i)}(), z' > \mu$ - null set

Proof

Since $< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E^{i}_{i-1}), z' >^{1\setminus p}$ is bounded for each n, then $< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E^{i}_{i-1}), z' >^{1\setminus p} \leq \beta$ for $\beta > 0$.

From the hypothesis, $M \leq f_n(x_{i-1},x_i)$ on $\Pi_{i=1}E_{n\,i-1}{}^{i}$. Therefore,

$M < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' > \leq < \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' >^{1\setminus p}.$

Since $\Pi_{i=1}E_{n\,i-1}{}^{i}$ is monotonically increasing to $\Pi_{i=1}E^{i}_{i-1}$, it follows that

$\Pi_{i=1}E_{n\,i-1}{}^{i} \uparrow \Pi_{i=1}E^{i}_{i-1}$ (Otanga and Oduor [6])

Therefore,

$M < \mu^{(i-1,i)}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' > \leq < \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}E_{n\,i-1}{}^{i}), z' >^{1\setminus p} \leq \beta$ for $\beta > 0$.

$LUB < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >=< \mu^{(i-1,i)}(\Pi_{i=1}E^{i}_{i-1}), z' >$.
Subsequently,

$< \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' > \leq \beta$

From $\Pi_{i=1}E^{i}_{i-1} = \cap_{n=1}^{\infty}\Pi_{i=1}E_{ni-1}{}^{i}$, we have $\Pi_{i=1}E_{ni-1}{}^{i} \downarrow \Pi_{i=1}E^{i}_{i-1}$.

This implies that $\Pi_{i=1}E^{i}_{i-1} \subset \Pi_{i=1}E_{ni-1}{}^{i}$ for $i = 1, 2, ...n$

Hence,

$< \mu^{(i-1,i)}(\Pi_{i=1}E^{i}_{i-1}), z' > \leq \beta \setminus M$

Taking $M \to \infty$, we obtain

$< \mu^{(i-1,i)}(\Pi_{i=1}E^{i}_{i-1}), z' >= 0$

4 Conclusion

The results obtained in this paper demonstrate utility of concepts of vector measure duality, continuity from below of a measure and monotonicity of a vector measure in integrating functions in $L_P(\mu^{(i-1,i)})$ for $0 < p < \infty$

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Campo R, Fernandez A, Ferrrando I, Mayoral F, Naranjo F. Compactness of multiplication operators on spaces of integrable functions with respect to a vector measure, in vector measure, integration and related topics. Birkhauser Verlag, Basel. 2010;201:109-113.

[2] Lech D, Iwo L. Topological vector spaces of Brochner measurable functions. Illinois j. Math. Spring. 2000:46:287-318.

[3] Okada S, Ricker WJ, Sanchez PEA. Optimal domain and integral extension of operators. Birkhauser, Basel. Switzerland. 2008;180:141-148.

[4] Otanga OL. On the finiteness and sigma finiteness of a cubic measure function. Inter. Math. Forum, Hikari Ltd. 2015;10(3):111-114.

[5] Otanga OL, Oduor MO, Aywa SO. Partition of measurable sets. Journal of Advances in Mathematics. 2015;10(8):3759-3763.

[6] Otanga OL, Oduor MO. On Pointwise product vector measure duality. Journal of Advances in Mathematics. 2021;20:8-18.

[7] Otanga OL. On Absolute continuity of non negative functions. Asian Research Journal of Mathematics. 2022;8(11):385-392.

[8] Rodriguez J. On integration of vector functions with respect to vector measures. Czechoslovak Math. J. 2006:56(3):805-825.

[9] Sanchez PEA. Vector measure duality and tensor product representation of L_p - spaces of vector measures. Amer. Math. Soc. 2004;132:3319-3326.

[10] Yaogan M. Facts about the fourier-stieltjes transform of vector measures on compact groups. Inter. J. Analy. Appl. 2013;19-25.

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