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ON THE REGULAR ELEMENTS OF RINGS IN WHICH
THE PRODUCT OF ANY TWO ZERO DIVISORS
LIES IN THE GALOIS SUBRING

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Abstract: Suppose R is a completely primary finite ring in which the product of any two zero divisors lies in the Galois (coefficient) subring. We construct R and find a generalized characterization of its regular elements.

AMS Subject Classification: 13M05, 16P10, 16U60, 13E10, 16N20

Key Words: unit groups, completely primary finite rings

1. Introduction

Unless otherwise stated, $J(R)$ shall denote the Jacobson radical of a completely primary finite ring R . We shall denote the coefficient (Galois) subring of R by R' . The set of all the regular elements in R shall be denoted by $V(R)$. The rest of the notations shall be adopted from [1].

An element $x \in R$ is called regular if there exists $y \in R$ such that $x = x^2y$. The element y is called a von Neumann inverse of x , see e.g [2]. It is well known that in any local ring, a regular element is either a unit or zero. Further details

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on the classes of completely primary finite rings considered in this work may be obtained in [3] and [4].

2. The Construction

Let R' be the Galois ring of the form $GR(p^{nr}, p^n)$. For each $i = 1, \dots, h$, let $u_i \in J(R)$, such that U is an h -dimensional R' -module generated by $\{u_1, \dots, u_h\}$ so that $R = R' \oplus U$ is an additive group. On this group, define multiplication by the following relations:

(i) If $n = 2$, then $u_i u_j = p\alpha_{ij}$, $u_i^3 = u_i^2 u_j = u_i u_j^2 = 0$, $u_i r' = (r')^{\sigma_i} u_i$

(ii) If $n \geq 3$, then

$$p^{n-1} u_i = 0, u_i u_j = p^2 \alpha_{ij} + p^{n-1} \beta_{ij}, u_i^n = u_i^{n-1} u_j = u_i u_j^{n-1} = 0, u_i r' = (r')^{\sigma_i} u_i,$$

where $r', \alpha_{ij} \in R'$, $\beta_{ij} \in R'/pR'$, $1 \leq i, j \leq h$ and σ_i is the automorphism associated with u_i . Further, let $pu_i = u_i u_j = 0$, when $u_i \in U$.

From the given multiplication in R , we notice that $r', s' \in R'$, $\gamma_i, \lambda_i \in F_0$ are elements of R , then

$$\begin{aligned} (r' + \sum_{i=1}^h \lambda_i u_i)(s' + \sum_{i=1}^h \lambda_i u_i) &= r' s' + p^{n-1} \sum_{i,j=1}^h \xi_{ij} (\lambda_i (\gamma_j)^{\sigma_i} + pR') \\ &\quad + \sum_{i=1}^h [(r' + pR') \gamma_i + \lambda_i (s' + pR')^{\sigma_i}] u_i, \end{aligned}$$

where $r', s' \in R'$, $\lambda_i, \gamma_i \in F_0$, $\xi_{ij} \in R'/pR'$. It is easy to verify that the given multiplication turns R into a ring with identity $(1, 0, \dots, 0)$. We also notice that $p^{n-1} \in (J(R))^2$ when $\text{char } R = \text{char } R' = p^n$, $n \geq 2$. Specifically, $p \in (J(R))^2$ when $n = 2$.

3. Preliminary Results

Lemma 1. *The ring described by the construction is commutative iff $\sigma_i = \text{id}_{R'}$ for each $i = 1, \dots, h$.*

Proof. It is evident □

Remark: If $n = 2$, then the construction yields rings satisfying the properties

$$J(R) = pR' \oplus U$$

$$(J(R))^2 = pR'$$

$$(J(R))^3 = (0).$$

On the other hand, if $n \geq 3$, then $J(R) = pR' \oplus U$

$$(J(R))^{n-1} = p^{n-1}R'$$

$$(J(R))^n = (0).$$

Now, consider a commutative ring R from the class of rings described by the construction, we notice that

$$R = R' \oplus \sum_{i=1}^h R' u_i$$

$$J(R) = pR' \oplus \sum_{i=1}^h R' u_i.$$

So

$$1 + J(R) = 1 + pR' \oplus \sum_{i=1}^h R' u_i.$$

Further, $V(R) = R^* \cup \{0\} = (R^*/1 + J(R)).(1 + J(R)) \cup \{0\} = \langle a \rangle .(1 + J(R)) \cup \{0\} \cong \langle a \rangle \times (1 + J(R)) \cup \{0\} \cong \mathbf{Z}_{p^r-1} \times (1 + J(R)) \cup \{0\}$. It therefore suffices to determine the structure of $1 + J(R)$.

Proposition 1. For each prime integer p , $1 + pR'$ is a subgroup of $1 + J(R)$.

Proposition 2. For each prime integer p , $1 + pR' \oplus R' u_1$ is a subgroup of $1 + J(R)$.

Proposition 3. For each $1 \leq j \leq h$, $1 + \sum_{j=1}^h R' u_j$ is a subgroup of $1 + J(R)$.

Since the two sided annihilator $\text{ann}(J(R)) = p^{n-1}R'$, we state the following result

Proposition 4. $1 + \text{ann}(J(R)) \leq 1 + pR' \leq 1 + J(R)$.

Proof. It suffices to prove that $1 + \text{ann}(J(R)) \leq 1 + pR'$. Clearly $1 + \text{ann}(J(R)) = 1 + p^{n-1}R'$, $\forall n \geq 2$. Now, for $r', s' \in R'$, let $1 + p^{n-1}r', 1 + p^{n-1}s' \in 1 + \text{ann}(J(R))$. Then

$$\begin{aligned} (1 + p^{n-1}r')(1 + p^{n-1}s')^{-1} &= (1 + p^{n-1}r')(1 - p^{n-1}s') \\ &= 1 + p^{n-1}(r' - s') \in 1 + \text{ann}(J(R)). \quad \square \end{aligned}$$

Proposition 5. *Let $p = 2$. Then the 2- group $1 + J(R)$ is a direct product of the subgroups $1 + pR' \oplus R'u_1$ by $1 + \sum_{i=1}^h \oplus R'u_i$, with $h \geq 2$.*

Proposition 6. *Let $p \neq 2$. The p - group $1 + J(R)$ is a direct product of the subgroups $1 + pR'$ by $1 + \sum_{i=1}^h \oplus R'u_i$.*

Proposition 7. *Let U be a finitely generated R' - module. If U is generated by $\{u_1, \dots, u_h\}$, then $\{u_1, u_1 + u_2, \dots, u_{h-1} + u_h\}$ also generates U .*

Proof. If U is a finitely generated R' - module, then there exist $\alpha_1, \dots, \alpha_h \in R'$, such that every $u \in U$ can be expressed in the form $u = \sum_{i=1}^h \alpha_i u_i$. But $\sum_{i=1}^h \alpha_i u_i = (\alpha_1 - \alpha_2 + \dots + (-1)^{h+1} \alpha_h)u_1 + (\alpha_2 - \alpha_3 + \dots + (-1)^h \alpha_h)(u_1 + u_2) + \dots + (\alpha_{h-1} - \alpha_h)(u_{h-2} + u_{h-1}) + \alpha_h(u_{h-1} + u_h)$. Since all the coefficients $\alpha_1 - \alpha_2 + \dots + (-1)^{h+1} \alpha_h, \alpha_2 - \alpha_3 + \dots + (-1)^h \alpha_h, \dots, \alpha_{h-1} - \alpha_h$ and α_h belong to R' , it follows that $\{u_1, u_1 + u_2, \dots, u_{h-1} + u_h\}$ generates U . $\square$

Proposition 8. *Let R be a commutative finite ring from the class of finite rings described by the construction. If U is generated by $\{u_1, \dots, u_h\}$, then it is also generated by $\{u_1, u_1 + u_2, \dots, u_1 + u_2 + \dots + u_h\}$.*

4. Main Results

Proposition 9. *Let R be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$ and $\text{char } R = p^2$, then*

$$1 + J(R) \cong \begin{cases} \mathbf{Z}_4' \times (\mathbf{Z}_2')^{h-1} & \text{if } p = 2 \\ \mathbf{Z}_p' \times (\mathbf{Z}_p')^h & \text{if } p \neq 2 \end{cases}$$

Proof. Let $\lambda_1, \dots, \lambda_r \in R'$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, \dots, \overline{\lambda_r} \in R'/pR'$ form a basis for R'/pR' regarded as a vector space over its prime subfield $GF(p)$. Since the two cases do not overlap, we treat them in turn.

Case (i). $p = 2$. We notice that, for every $\nu = 1, \dots, r$ and $u_1 \in J(R) - (J(R))^2$,

$$(1 + \lambda_\nu u_1)^2 = 1 + 2\lambda_\nu^2 + 2\lambda_\nu u_1$$

$$= 1 + 2\lambda_\nu^2, \text{ since } 2 \in (J(R))^2 \text{ and } 2u_1 = 0.$$

Now,

$$\begin{aligned} (1 + 2\lambda_\nu^2)(1 + \lambda_\nu u_1) &= 1 + 2\lambda_\nu^2 + (\lambda_\nu + 2\lambda_\nu^3)u_1 \\ &= 1 + 2\lambda_\nu^2 + \lambda_\nu u_1, \text{ since } 2 \in (J(R))^2 \text{ and } 2u_1 = 0. \end{aligned}$$

But then,

$$\begin{aligned} (1 + 2\lambda_\nu^2 + \lambda_\nu u_1)(1 + \lambda_\nu u_1) &= 1 + 2^2\lambda_\nu^2 + 2(\lambda_\nu + \lambda_\nu^3)u_1 \\ &= 1, \text{ since } 2 \in (J(R))^2 \text{ and } 2u_1 = 0. \end{aligned}$$

Also, for each $u_i \in J(R) - (J(R))^2$, $1 \leq i \leq h-1$, $(1 + \lambda_\nu u_i + \lambda_\nu u_{i+1})^2 = 1 + 2(2^2\lambda_\nu^2) + 2\lambda_\nu(u_i + u_{i+1}) = 1$, since $(J(R))^3 = (0)$ so that $2^3 = 0$, $2u_i = 2u_{i+1} = 0$, as $2 \in (J(R))^2$.

So, for each $\nu = 1, \dots, r$ and $1 \leq i \leq h-1$, $(1 + \lambda_\nu u_1)^4 = 1$, $(1 + \sum_{i=1}^{h-1} \lambda_\nu(u_i + u_{i+1}))^2 = 1$.

For positive integers α_ν , $\beta_{i\nu}$ with $\alpha_\nu \leq 4$, $\beta_{i\nu} \leq 2$ ($1 \leq i \leq h-1$, $1 \leq \nu \leq r$), we notice that the equation

$$\prod_{\nu=1}^r \{(1 + \lambda_\nu u_1)^{\alpha_\nu}\} \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\beta_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = 4$ and $\beta_{i\nu} = 2$, $1 \leq i \leq h-1$. If we set

$$T_\nu = \{(1 + \lambda_\nu u_1)^\alpha \mid \alpha = 1, \dots, 4\},$$

$$S_{i\nu} = \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\beta_i} \mid \beta_i = 1, 2\}$$

we see that T_ν , $S_{i\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^r |< 1 + \lambda_\nu u_1 >| \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r |< 1 + \lambda_\nu(u_i + u_{i+1}) >| = 2^{(h+1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the hr subgroups T_ν , $S_{i\nu}$, $1 \leq i \leq h-1$ is direct. Therefore, their product exhausts the group $1 + J(R)$.

Case (ii). p is odd. If $\nu = 1, \dots, r$ and $u_i \in J(R) - (J(R))^2$, $1 \leq i \leq h-1$,

$$(1 + p\lambda_\nu)^p = 1 + p^2\lambda_\nu + \frac{p(p-1)}{2}(p\lambda_\nu)^2 + \dots + (p\lambda_\nu)^p$$

$$= 1, \text{ since } \text{char} R = p^2.$$

Also,

$$(1 + \lambda_\nu u_1)^p = (1 + \sum_{i=1}^2 \lambda_\nu u_i)^p = \dots = (1 + \sum_{i=1}^h \lambda_\nu u_i)^p = 1.$$

For positive integers α_ν , $\beta_{i\nu}$ with $\alpha_\nu \leq p$, $\beta_{i\nu} \leq p$ ($1 \leq i \leq h, 1 \leq \nu \leq r$), we notice that the equation

$$\prod_{\nu=1}^r \{(1 + p\lambda_\nu)^{\alpha_\nu}\} \cdot \prod_{i=1}^h \prod_{\nu=1}^r \{(1 + \sum_{j=1}^i \lambda_\nu u_j)^{\beta_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = \beta_{i\nu} = p, 1 \leq i \leq h$. If we set

$$T_\nu = \{(1 + p\lambda_\nu)^\alpha \mid \alpha = 1, \dots, p\},$$

$$S_{i\nu} = \{(1 + \sum_{j=1}^i \lambda_\nu u_j)^{\beta_i} \mid \beta_i = 1, \dots, p\}$$

we see that $T_\nu, S_{i\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^r |< 1 + p\lambda_\nu >| \cdot \prod_{i=1}^h \prod_{\nu=1}^r |< 1 + \sum_{j=1}^i \lambda_\nu u_j >| = p^{(h+1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups $T_\nu, S_{i\nu}, 1 \leq i \leq h$ is direct. Therefore, their product exhausts the group $1 + J(R)$. $\square$

Proposition 10. *Let R be a commutative finite ring from the class of finite rings given by the construction. If $h \geq 1, r > 1$ and $\text{char} R = p^3$, then*

$$1 + J(R) \cong \begin{cases} \mathbf{Z}_2^r \times \mathbf{Z}_4^r \times (\mathbf{Z}_2^r)^{h-1} & \text{if } p = 2 \\ \mathbf{Z}_{p^2}^r \times (\mathbf{Z}_p^r)^h & \text{if } p \neq 2 \end{cases}$$

Proof. Let $\lambda_1, \dots, \lambda_r \in R'$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, \dots, \overline{\lambda_r} \in R'/pR'$ form a basis for R'/pR' regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

Case (i). $p = 2$. We notice that for every $\nu = 1, \dots, r$ and $u_1 \in J(R) - (J(R))^2$,

$$(-1 + 4\lambda_\nu)^2 = 1 - 2^3\lambda_\nu + 2^4\lambda_\nu^2$$

$$= 1, \text{ since } \text{char} R = 2^3.$$

Also

$$\begin{aligned} (1 + \lambda_\nu u_1)^2 &= 1 + 2^2 \lambda_\nu^2 + 2 \lambda_\nu u_1 \\ &= 1 + 2^2 \lambda_\nu^2, \text{ since } 2u_1 = 0. \end{aligned}$$

But then,

$$(1 + 2^2 \lambda_\nu^2)^2 = 1 + 2^3 \lambda_\nu^2 + 2^4 \lambda_\nu^4$$

$$= 1, \text{ since } \text{char} R = 2^3.$$

It is also easy to see that, for each $\nu = 1, \dots, r$, $1 \leq i \leq h-1$, $(1 + \lambda_\nu(u_i + u_{i+1}))^2 = 1$.

For positive integers α_ν , β_ν , $\kappa_{i\nu}$ with $\alpha_\nu \leq 2$, $\beta_\nu \leq 4$, $\kappa_{i\nu} \leq 2$, ($1 \leq i \leq h-1$, $1 \leq \nu \leq r$), we notice that the equation

$$\prod_{\nu=1}^r \{(-1 + 4\lambda_\nu)^{\alpha_\nu}\} \cdot \prod_{\nu=1}^r \{(1 + \lambda_\nu u_1)^{\beta_\nu}\} \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\kappa_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = 2$ and $\beta_\nu = 4$, $\kappa_{i\nu} = 2$, $1 \leq i \leq h-1$. If we set

$$H_\nu = \{(-1 + 4\lambda_\nu)^\alpha \mid \alpha = 1, 2\},$$

$$T_\nu = \{(1 + \lambda_\nu u_1)^\beta \mid \beta = 1, \dots, 4\},$$

$$S_{i\nu} = \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\kappa_i} \mid \kappa_i = 1, 2\}$$

we see that H_ν , T_ν , $S_{i\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^r |< -1 + 4\lambda_\nu >| \cdot \prod_{\nu=1}^r |< 1 + \lambda_\nu u_1 >| \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r |< 1 + \lambda_\nu(u_i + u_{i+1}) >| = 2^{(h+2)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups H_ν , T_ν , $S_{i\nu}$, $1 \leq i \leq h-1$ is direct. Therefore, their product exhausts the group $1 + J(R)$.

Case (ii). p is odd. Here, we notice that

$$(1 + p\lambda_\nu)^{p^2} = 1, (1 + \lambda_\nu u_1)^p = (1 + \sum_{i=1}^2 \lambda_\nu u_i)^p = \dots = (1 + \sum_{i=1}^h \lambda_\nu u_i)^p = 1.$$

Now, for positive integers α_ν , $\beta_{i\nu}$ with $\alpha_\nu \leq p^2$, $\beta_{i\nu} \leq p$, ($1 \leq i \leq h, 1 \leq \nu \leq r$), we notice that the equation

$$\prod_{\nu=1}^r \{(1 + p\lambda_\nu)^{\alpha_\nu}\} \cdot \prod_{i=1}^h \prod_{\nu=1}^r \{(1 + \sum_{j=1}^i \lambda_\nu u_j)^{\beta_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = p^2$, $\beta_{i\nu} = p$ for $1 \leq \nu \leq r$ and $1 \leq i \leq h$. The rest of the proof is similar to Case (ii) in the previous proposition.

Proposition 11. *Let R be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r = 1$ and $\text{char} R = p^n$, $n \geq 4$, then*

$$1 + J(R) \cong \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times (\mathbf{Z}_2)^{h-1} & \text{if } p = 2 \\ \mathbf{Z}_{p^{n-1}} \times (\mathbf{Z}_p)^h & \text{if } p \neq 2 \end{cases}$$

Proof. Case (i). $p = 2$. Consider the element $1 + 2t + u_1$, where $t = n - 4$, $n \geq 4$, then $o(1 + 2t + u_1) = 2^{n-2}$. The elements $-1 + 2^{n-1}$ and $-1 + 2^{n-2} + u_1$ are each of order 2. Also, the elements $1 + u_1 + u_2$, $1 + u_2 + u_3, \dots, 1 + u_{h-1} + u_h$ are each of order 2. Now, the mentioned elements generate cyclic subgroups of $1 + J(R)$. Since $|\langle 1 + 2t + u_1 \rangle| \cdot |\langle -1 + 2^{n-1} \rangle| \cdot |\langle -1 + 2^{n-2} + u_1 \rangle| \cdot \prod_{j=2}^h |\langle 1 + u_{j-1} + u_j \rangle| = 2^{n+h-1}$, and the intersection of any pair of the cyclic subgroups gives the identity group, $\langle 1 + 2t + u_1 \rangle \times \langle -1 + 2^{n-1} \rangle \times \langle -1 + 2^{n-2} + u_1 \rangle \times \langle 1 + u_1 + u_2 \rangle \times \dots \times \langle 1 + u_{h-1} + u_h \rangle$ is a direct product.

Case (ii). $p \neq 2$ Here, the element $1 + p$ is of order p^{n-1} while the elements $1 + u_1, 1 + \sum_{i=1}^2 u_i, \dots, 1 + \sum_{i=1}^h u_i$ are each of order p . The given elements generate cyclic subgroups of the group $1 + J(R)$. Since

$$|\langle 1 + p \rangle| \cdot \prod_{i=1}^h |\langle 1 + \sum_{i=1}^i u_i \rangle| = p^{n+h-1},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, $\langle 1 + p \rangle \times \langle 1 + u_1 \rangle \times \langle 1 + \sum_{i=1}^2 u_i \rangle \times \dots \times \langle 1 + \sum_{i=1}^h u_i \rangle$ is a direct product. $\square$

Proposition 12. *Let R be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r > 1$ and $\text{char} R = p^4$, then*

$$1 + J(R) \cong \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4 \times \mathbf{Z}_2^{r-1} \times \mathbf{Z}_8^{r-1} \times (\mathbf{Z}_2^r)^{h-1} & \text{if } p = 2 \\ \mathbf{Z}_{p^3} \times (\mathbf{Z}_p^r)^h & \text{if } p \neq 2 \end{cases}$$

Let $\lambda_1, \dots, \lambda_r \in R'$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, \dots, \overline{\lambda_r} \in R'/pR'$ form a basis for R'/pR' regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

Case (i). $p = 2$. Clearly,

$$\begin{aligned} (-1 + 2^3\lambda_1)^2 &= 1, (-1 + 2^2\lambda_1 + \lambda_1u_1)^2 = 1, (-1 + 2^3(\lambda_1 + \lambda_2) + \lambda_2u_1)^4 = 1, \\ (1 + 2^2(\lambda_1 + \lambda_2) + \lambda_2u_1)^2 &= (1 + 2^2(\lambda_1 + \lambda_3) + (\lambda_2 + \lambda_3)u_1)^2 = \dots = \\ (1 + 2^2(\lambda_1 + \lambda_r) + (\lambda_2 + \dots + \lambda_r)u_1)^2 &= 1, (1 + 2\lambda_\nu + \lambda_\nu u_1)^8 = 1, \\ (1 + \lambda_\nu u_{j-1} + \lambda_\nu u_j)^2 &= 1, \quad 2 \leq j \leq h. \end{aligned}$$

For positive integers $\alpha, \beta, \kappa, \gamma_s, \tau_\nu, \omega_{i\nu}$ with $\alpha \leq 2, \beta \leq 2, \kappa \leq 4, \gamma_s \leq 2, \tau_\nu \leq 8, \omega_{i\nu} \leq 2, 2 \leq s \leq r, 1 \leq \nu \leq r, 1 \leq i \leq h-1$, we notice that the equation $\{(-1 + 2^3\lambda_1)^\alpha\} \cdot \{(-1 + 2^2\lambda_1 + \lambda_1u_1)^\beta\} \cdot \{(-1 + 2^3(\lambda_1 + \lambda_2) + \lambda_2u_1)^\kappa\} \cdot \prod_{\nu=2}^r \{(1 + 2^2(\lambda_1 + \lambda_\nu) + \sum_{i=2}^\nu \lambda_i u_i)^{\gamma_\nu}\} \cdot \prod \{(1 + 2\lambda_\nu + \lambda_\nu u_1)^{\tau_\nu}\} \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\omega_{i\nu}}\} = \{1\}$, will imply $\alpha = \beta = 2, \kappa = 4, \gamma_\nu = 2, \tau_\nu = 8, \omega_{i\nu} = 2$ for every $\nu = 1, \dots, r, \nu = 2, \dots, r$ and $i = 1, \dots, h-1$. If we set

$$E = \{(-1 + 2^3\lambda_1)^\alpha \mid \alpha = 1, 2\},$$

$$F = \{(-1 + 2^2\lambda_1 + \lambda_1u_1)^\beta \mid \beta = 1, 2\},$$

$$G = \{(-1 + 2^3(\lambda_1 + \lambda_2) + \lambda_2u_1)^\kappa \mid \kappa = 1, \dots, 4\},$$

$$H_\nu = (1 + 2^2(\lambda_1 + \lambda_\nu) + \sum_{i=2}^\nu \lambda_i u_i)^{\gamma_\nu} \mid \gamma_\nu = 1, 2\},$$

$$K_\nu = \{(1 + 2\lambda_\nu + \lambda_\nu u_1)^{\tau_\nu} \mid 1 \leq \tau_\nu \leq 8\},$$

$$L_{i\nu} = \{(1 + \lambda_\nu(u_i + u_{i+1}))^{\omega_{i\nu}}\}$$

we see that $E, F, G, H_2, \dots, H_r, K_2, \dots, K_r, L_{1\nu}, \dots, L_{(h-1)\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition.

Since

$$|< -1 + 8\lambda_1 >| \cdot |< -1 + 4\lambda_1 + \lambda_1u_1 >| \cdot |< -1 + 8(\lambda_1 + \lambda_2) + \lambda_2u_1 >| \cdot$$

$$\prod_{\nu=2}^r |< 1 + 4(\lambda_1 + \lambda_\nu) + \sum_{i=2}^\nu \lambda_i u_i >| \cdot \prod_{\nu=2}^r |< 1 + 2\lambda_\nu + \lambda_\nu u_1 >| \cdot$$

$$\prod_{i=1}^{h-1} \prod_{\nu=1}^r |< 1 + \lambda_\nu(u_i + u_{i+1}) >| = 2^{(h+1)r},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $1 + (h + 1)r$ subgroups

$$E, F, G, H_2, \dots, H_r, K_2, \dots, K_r, L_{1\nu}, \dots, L_{(h-1)\nu}$$

is direct. Therefore, their product exhausts $1 + J(R)$.

Case (ii). $p \neq 2$. Here the proof is similar to that of Case (ii) in the previous proposition, with some slight modification. $\square$

Proposition 13. *Let R be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r > 1$ and $\text{char} R = p^n$, $n \geq 5$, then*

$$1 + J(R) \cong \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times \mathbf{Z}_{2^{n-3}}^{r-1} \times \mathbf{Z}_8^{r-1} \times (\mathbf{Z}_2^r)^{h-1} & \text{if } p = 2 \\ \mathbf{Z}_{p^{n-1}}^r \times (\mathbf{Z}_p^r)^h & \text{if } p \neq 2 \end{cases}$$

Proof. Let $\lambda_1, \dots, \lambda_r \in R'$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, \dots, \overline{\lambda_r} \in R'/pR'$ form a basis for R'/pR' regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

Case (i). $p = 2$. Clearly,

$$(-1 + 2^{n-1}\lambda_1)^2 = 1, (-1 + 2^{n-1}\lambda_1 + 2^{n-1}\lambda_2)^2 = 1, (1 + 2\lambda_1 + \lambda_1 u_1)^{2^{n-2}} = 1,$$

$(1 + \sum_{i=2}^{\nu} \lambda_i u_i)^{2^{n-3}} = 1, (1 + 4\lambda_{\nu} + \lambda_{\nu} u_1)^{2^{n-2}} = 1, \nu = 2, \dots, r, (1 + \lambda_{\nu}(u_i + u_{i+1}))^2 = 1, 1 \leq i \leq h-1$ For positive integers $\alpha, \beta, \kappa_{\nu}, \gamma_{\nu}, \tau_{\nu}, \omega_{i\nu}$ with $\alpha \leq 2, \beta \leq 2^{n-2}, \kappa_{\nu} \leq 2, \gamma_{\nu} \leq 2^{n-3}, \tau_{\nu} \leq 2^{n-2}, \omega_{i\nu} \leq 2, 1 \leq \nu \leq r, 1 \leq \nu \leq r, 1 \leq i \leq h-1$ we notice that the equation $\{(-1 + 2^{n-1}\lambda_1)^{\alpha}\} \cdot \{(1 + 2\lambda_1 + \lambda_1 u_1)^{\beta}\} \cdot \{(-1 + 2^{n-1} \sum_{\nu=1}^2 \lambda_{\nu})^{\kappa}\} \cdot \prod_{\nu=2}^r \{(1 + \sum_{i=2}^{\nu} \lambda_i u_i)^{\gamma_{\nu}}\} \cdot \prod \{(1 + 4\lambda_{\nu} + \lambda_{\nu} u_1)^{\tau_{\nu}}\} \cdot \prod_{i=1}^{h-1} \prod_{\nu=1}^r \{(1 + \lambda_{\nu}(u_i + u_{i+1}))^{\omega_{i\nu}}\} = \{1\}$, will imply $\alpha = 2, \beta = 2^{n-2}, \kappa = 2, \gamma_{\nu} = 2^{n-3}, \tau_{\nu} = 2^{n-2}, \omega_{i\nu} = 2$ for every $\nu = 2, \dots, r$ and $i = 1, \dots, h-1$. If we set

$$E = \{(-1 + 2^{n-1}\lambda_1)^{\alpha} \mid \alpha = 1, 2\}$$

$$F = \{(1 + 2\lambda_1 + \lambda_1 u_1)^{\beta} \mid \beta = 1, \dots, 2^{n-2}\}$$

$$G = \{(-1 + 2^{n-1}(\lambda_1 + \lambda_2))^{\kappa} \mid \kappa = 1, 2\}$$

$$H_{\nu} = \{(1 + \sum_{i=2}^{\nu} \lambda_i u_i)^{\gamma_{\nu}} \mid \gamma_{\nu} = 1, \dots, 2^{n-3}\},$$

$$K_{\nu} = \{(1 + 4\lambda_{\nu} + \lambda_{\nu} u_1)^{\tau_{\nu}} \mid 1, \dots, 2^{n-2}\},$$

$$L_{i\nu} = \{(1 + \lambda_{\nu}(u_i + u_{i+1}))^{\omega_i} \mid \omega_i = 1, 2\}$$

we see that $E, F, G, H_2, \dots, H_r, K_2, \dots, K_r, L_{1\nu}, \dots, L_{(h-1)\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition.

Since

$$|< -1 + 2^{n-1}\lambda_1 >| \cdot |< 1 + 2\lambda_1 + \lambda_1 u_1 >| \cdot \prod_{\nu=2}^r |< -1 + 2^{n-1}(\lambda_1 + \lambda_2) >| \cdot$$

$$\prod_{\nu=2}^r |< 1 + \sum_{\iota=2}^{\nu} \lambda_{\iota} u_1 >| \cdot \prod_{\nu=2}^r |< 1 + 4\lambda_{\nu} + \lambda_{\nu} u_1 >| \cdot$$

$$\prod_{i=1}^{h-1} \prod_{\nu=1}^r |< 1 + \lambda_{\nu}(u_i + u_{i+1}) >| = 2^{(h+n-1)r},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $1 + (h+1)r$ subgroups $E, F, G, H_2, \dots, H_r, K_2, \dots, K_r, L_{1\nu}, \dots, L_{(h-1)\nu}$ is direct. Therefore, their product exhausts $1 + J(R)$.

Case (ii). $p \neq 2$. Here the proof is similar to that of Case (ii) in the previous proposition, with some slight modification. $\square$

We now state the main result.

Theorem 1. *The regular elements of the rings described by the construction is given as follows:*

i) *If $\text{char} R = p^2$, then*

$$V(R) \cong \begin{cases} \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_4^r \times (\mathbf{Z}_2^r)^{h-1} \cup \{0\} & \text{if } p = 2 \\ \mathbf{Z}_{p^{r-1}} \times \mathbf{Z}_p^r \times (\mathbf{Z}_p^r)^h \cup \{0\} & \text{if } p \neq 2 \end{cases}$$

ii) *If $\text{char} R = p^3$, then*

$$V(R) \cong \begin{cases} \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_2^r \times \mathbf{Z}_4^r \times (\mathbf{Z}_2^r)^{h-1} \cup \{0\} & \text{if } p = 2 \\ \mathbf{Z}_{p^{r-1}} \times \mathbf{Z}_{p^2}^r \times (\mathbf{Z}_p^r)^h \cup \{0\} & \text{if } p \neq 2 \end{cases}$$

iii) *If $\text{char} R = p^4$, then*

$$V(R) \cong \begin{cases} \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4 \times (\mathbf{Z}_2^r)^{h-1} \cup \{0\} & \text{if } p = 2 \text{ and } r = 1 \\ \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4 \times \mathbf{Z}_2^{r-1} \times \mathbf{Z}_8^{r-1} \times (\mathbf{Z}_2^r)^{h-1} \cup \{0\} & \text{if } p = 2 \text{ and } r > 1 \\ \mathbf{Z}_{p^{r-1}} \times \mathbf{Z}_{p^3}^r \times (\mathbf{Z}_p^r)^h \cup \{0\} & \text{if } p \neq 2 \end{cases}$$

iv) If $\text{char } R = p^n$, $n \geq 5$, then

$$V(R) \cong \begin{cases} \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times (\mathbf{Z}_2)^{h-1} \cup \{0\} & \text{if } p = 2 \text{ and } r = 1 \\ \mathbf{Z}_{2^{r-1}} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times \mathbf{Z}_{2^{n-3}}^{r-1} \times \mathbf{Z}_{2^{n-2}}^{r-1} \times (\mathbf{Z}_2)^{h-1} \cup \{0\} & \text{if } p = 2 \text{ and } r > 1 \\ \mathbf{Z}_{p^{r-1}} \times \mathbf{Z}_{p^{n-1}} \times (\mathbf{Z}_p^r)^h \cup \{0\} & \text{if } p \neq 2 \end{cases}$$

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References

- [1] C.J. Chikunji, Unit groups of cube radical zero commutative completely primary finite rings, *International Journal of Mathematics and Mathematical Sciences*, **2005**, No. 4 (2005), 579-592.
- [2] A. Osama, A.O. Emad, On the regular elements in $\mathbf{Z}_n$, *Turkish Journal of Mathematics*, **32** (2008), 31-39.
- [3] R. Raghavendran, Finite associative rings, *Compositio Math.*, **21**, No. 2 (1969), 195-229.
- [4] Y. Alkhamees, Finite completely primary rings in which the product of any two zero divisors of a ring is in its coefficient subring, *International Journal of Mathematics and Mathematical Sciences*, **17**, No. 3 (1994), 463-468.