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The Wiener Index and Related Indices of the Zero Divisor Graphs of Certain Classes of Completely Primary Finite Rings

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Abstract

Studies on zero divisor graphs of completely primary finite rings R have been extensively done from Galois rings to other rings whose sets of zero divisors $Z(R)$ coincides with the Jacobson radical $J(R)$. The studies have focused on graph geometric properties such as the girth, clique number, chromatic number and diameter among others. Some findings are also evident on matrices of zero divisor graphs on certain classes of rings. The classes of completely primary finite rings considered in the various studies are square radical zero, cube radical zero and power four radical zero. In this paper we have advanced the study on zero divisor graph $\Gamma(R)$ of completely primary finite rings by investigating the Wiener Index and its invariants such as the Average disorder number and Distance index. Further, we analyse the binding number and some bounds on the Zagreb indices of the rings satisfying the conditions $(Z(R))^3 = (0)$ and $(Z(R))^2 \neq (0)$, $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$.

Keywords: Completely primary finite rings; Zero divisor graphs; Wiener index; Average disorder number; Distance index.

1 Introduction

In the entire paper, R represents cube radical zero or power four radical zero completely primary finite ring, the zero divisor graph of R denoted by $\Gamma(R)$ and $R' = GR(p^{kr}, p^k)$ to represent the Galois ring of order p^{kr} and characteristic p^k for some positive integers k, r . $Z(R)$ and $J(R)$ to represent the subsets of zero divisors and Jacobson radical respectively. Other notations are standard unless otherwise stated.

The concept on zero divisor graphs of commutative rings was introduced by Beck in [2] who centered his study on the colouration of the graphs and obtaining the chromatic number. All zero divisors formed the vertices of the graph. The graph was denoted by $G(R)$. An extension of their research was done by Anderson and Livingston in [1] which considered the vertices of the zero divisor graph $\Gamma(R)$ using the nonzero-zero divisors only. Their illustration of $\Gamma(R)$ was considered to be better in terms of graph representation of zero divisors. Their study investigated the graph geometric properties like the girth, diameter, chromatic number

among others.

Further, Walwenda *et al* in [15] investigated the zero divisor graphs of finite rings in which the product of two zero divisors lies in the coefficient subring where $\text{char}(R) = p^k, k \geq 2$ and $(Z(R))^2 \subseteq GR(p^{kr}, p^k)$ satisfying $Z(R) = pR' \oplus U$, $(Z(R))^{k-1} = p^{k-1}R'$ and $(Z(R))^k = (0)$. The research focussed on the geometric properties of $\Gamma(R)$ such as the diameter, girth and the binding number index.

Some studies have also been performed on graph indices such as the Wiener index $W(G)$ and its invariants such as the average disorder number $A(G)$ and the average distance $\mu(G)$. Some results are also available on the Zagreb indices $Z(G)$. The Wiener index is a graph invariant that belongs to the molecules structure -descriptors called topological indices and are used for the design of molecules with desired property [14]. In [10], a research was conducted on the Wiener index of the graph G and their line graph $L(G)$. They demonstrated that if G is of minimum degree at least 2, then $W(G) \leq W(L(G))$. It was further illustrated that for any nonnegative integer g_0 , there exist $g \geq g_0$ such that there are finitely many graphs G of girth g satisfying $W(G) = W(L(G))$. All the graphs considered were simple and undirected.

In chemical graph theory, computations for Wiener indices for cyclic carbon-chained organic compounds and its applications cannot be underestimated. This index has been quite handy in determination of the boiling points and polarity number of alkanes and their branched isomers. Further, the most and common natural field in the application of the Wiener index is the quantitative structure relationships especially in the estimation of emission spectra of the ultra violet radiations of α and β -unsaturated Ketone.

Another index, the Zagreb index was introduced in [7] and given an elaboration in [8]. Fundamental properties of the indices were given a summary in [11]. Other than the Wiener index $W(\Gamma(R))$, average disorder number $A(\Gamma(R))$, the average distance index $\mu(\Gamma(R))$ and the binding number $b(\Gamma(R))$ of the zero divisor graph $\Gamma(R)$, we have also obtained the bounds on the first and second Zagreb indices $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ respectively of the cube radical zero and power four radical zero completely primary finite rings R . For the bounds on the Zagreb indices, the invariants considered are the maximum degree $\Delta(\Gamma(R))$ and the minimum degree $\delta(\Gamma(R))$.

The constructions for the cube radical zero and power four radical zero completely primary finite rings considered in this paper can be obtained from [4] and [13] respectively.

2 Preliminaries

In this section, we provide some results and standard definitions which are useful in the sequel.

Definition 1. The binding number of $\Gamma(R)$ denoted by $b(\Gamma(R)) = \frac{|N(S)|}{|S|}$ where $S \subseteq V(\Gamma(R))$, $N(S) \neq V(\Gamma(R))$, $S \neq \emptyset$ such that

- (i) $N(S) \cup S = V(\Gamma(R))$.
- (ii) $N(S) \cap S = \emptyset$.
- (iii) $\deg(u) \leq \deg(v) \forall u \in S, v \in N(S)$.
- (iv) No pair of vertices in S are adjacent.

Lemma 1. [6] Let G be a graph of order n with m edges then

$$\sum_{i=1}^m d_i^2 \leq m \left[\frac{2m}{n-1} + n - 1 \right].$$

Lemma 2. [3] For positive real numbers $a_1, a_2, \dots, a_r, A^{\frac{1}{2}} \geq B^{\frac{1}{r-1}}$ where

$$A = \frac{2}{r(r-1)}(a_1a_2 + a_1a_3 + \dots + a_1a_r + a_2a_3 + \dots + a_{r-1}a_r)$$

$$B = \frac{1}{r}(a_1a_2 \dots a_{r-1} + a_1a_2 \dots a_{r-2}a_r \dots + a_2a_3 \dots a_{r-1}a_r).$$

Lemma 3. [9] Let $(a) = (a_1, a_2, \dots, a_r), (b) = (b_1, b_2, \dots, b_r)$ be two real r -tuples. Then

$$\sum_{i=1}^r a_i^2 \sum_{j=1}^r b_j^2 - \left(\sum_{i=1}^r a_i b_i \right)^2 = \sum_{1 \leq i < j \leq r} (a_i b_j - a_j b_i)^2.$$

3 Constructions

3.1 Construction I

We provide the general construction of the classes of the cube radical zero completely primary finite ring R . This construction can be obtained in [4].

For any prime integer p and a positive integer r , let $R' = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and of characteristic p^k and consider the annihilator of $Z(R)$ be $(Z(R))^2$ so that $R = R' \oplus U \oplus V$ is an additive abelian group where U and V are finitely generated R' -modules. Let s, t be non negative numbers of elements in the generating sets $\{u_1, u_2, \dots, u_s\}$ and $\{v_1, v_2, \dots, v_t\}$ for U and V respectively. Consider $t = \frac{s(s+1)}{2}$ for a fixed s and let $u_1, u_2, \dots, u_s$ be commuting indeterminates over Galois ring $R' = GR(p^{kr}, p^k)$ where $1 \leq k \leq 3$. Then

$R = R' \oplus \sum_{i=1}^s R' u_i \oplus \sum_{i,j=1}^s R' u_i u_j$. The multiplication in R is given by the following relations:

$$u_i u_j = u_j u_i = a_{ij}^k v, \quad u_i^2 = a_{ii}^k v, \quad u_i^3 = u_i^2 u_j = u_i u_j^2 = 0, \quad 1 \leq i, j \leq s$$

where (a_{ij}^k) defined in the multiplication of R is t -linearly independent matrices of dimension $s \times s$ with 1's in the $(i, j)^{th}$ and $(j, i)^{th}$ positions and 0-elsewhere.

If $x_o + \sum_{i=1}^s x_i u_i + \sum_{i,j=1}^s x_{ij} u_i u_j$ and $y_o + \sum_{i=1}^s y_i u_i + \sum_{i,j=1}^s y_{ij} u_i u_j$ are any two elements in R where $x_o, y_o \in R', x_i, y_i, x_{ij}, y_{ij} \in R'/pR'$ then from the multiplication defined on R ,

$$(x_o + \sum_{i=1}^s x_i u_i + \sum_{i,j=1}^s x_{ij} u_i u_j)(y_o + \sum_{i=1}^s y_i u_i + \sum_{i,j=1}^s y_{ij} u_i u_j) =$$

$$x_o y_o + \sum_{i=1}^s ((x_o + pR') y_i + x_i (y_o + pR')^{\sigma_i}) u_i + \sum_{i,j=1}^s (x_o y_{ij} + x_j (y_o)^{\sigma_i} + \sum_{i,j=1}^s a_{ij}^k x_i (y_j)^{\sigma_i}) u_i u_j$$

where σ_i is an identity automorphism in R' . From [4], the multiplication turns R into a commutative ring with identity $(1, 0, \dots, 0, 0, \dots, 0)$.

3.2 Construction II

The following construction describes the power four radical zero rings.

Let $R' = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k where $1 \leq k \leq 4$. Consider finitely

generated R' -modules U , V , and W such that $\dim_{R'} U = s$, $\dim_{R'} V = t$ and $\dim_{R'} W = \lambda$ and $s + t + \lambda = h$. Let the R' modules be generated by $\{u_1, u_2, \dots, u_s\}$, $\{v_1, v_2, \dots, v_t\}$ and $\{w_1, w_2, \dots, w_\lambda\}$ respectively so that $R = R' \oplus U \oplus V \oplus W$ is an additive abelian group. Suppose $s = 1$, $t = 1$ and $\lambda = h - 2$, then for $\text{char}(R) = p$, $R = R' \oplus R'u \oplus R'v \oplus \sum_{k=1}^{h-2} R'w_k$ where $pu = 0$, $p v = 0$, $p w_k = 0$ such that $1 \leq k \leq h - 2$ for any prime integer p . We define multiplication on R as follows;

$$(a_o, a_1, a_2, \dots, a_h)(b_o, b_1, b_2, \dots, b_h) = (a_o b_o, a_o b_1 + a_1 b_o, a_o b_2 + a_2 b_o + a_1 b_1, a_o b_3 + a_3 b_o + a_1 b_2 + a_2 b_1, \dots, a_o b_h + a_h b_o + a_1 b_2 + a_2 b_1).$$

For $\text{char}(R) = p^2$, assume $s = h - 1$, $t = 1$ and $\lambda = 0$ so that , $R = R' \oplus \sum_{i=1}^{h-1} R'u_i \oplus R'v$ where $pu_i \neq 0$, $p^2 u_i = 0$ and $p v = 0$ with $1 \leq i \leq s$. The following defines multiplication on R .

$$(a_o, a_1, a_2, \dots, a_{h-1}, \bar{a}_h)(b_o, b_1, b_2, \dots, b_{h-1}, \bar{b}_h) = (a_o b_o + p \sum_{i,j=1}^{h-1} a_i b_j, a_o b_1 + a_1 b_o, \dots, a_o b_{h-1} + a_{h-1} b_o, a_o \bar{b}_h + \bar{a}_h b_o)$$

where $\bar{a}_h, \bar{b}_h \in R'/pR'$.

For $\text{char}(R) = p^3$, consider $s = h - 1$, $t = 1$ and $\lambda = 0$ so that $R = R' \oplus \sum_{i=1}^{h-1} R'u_i \oplus R'v$ where $p^2 u_i \neq 0$, $p^3 u_i = 0$ where $1 \leq i \leq s$ and $p v = 0$. The following multiplication is defined on R :

$$(a_o, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_{h-1}, \tilde{a}_h)(b_o, \bar{b}_1, \bar{b}_2, \dots, \bar{b}_{h-1}, \tilde{b}_h) = (a_o b_o, a_o \bar{b}_1 + \bar{a}_1 b_o, \dots, a_o \bar{b}_{h-1} + \bar{a}_{h-1} b_o, a_o \tilde{b}_h + \tilde{a}_h b_o + \sum_{i,j=1}^{h-1} \bar{a}_i \bar{b}_j)$$

where $\bar{a}_i, \bar{b}_j \in R'/p^2 R'$ and $\tilde{a}_h, \tilde{b}_h \in R'/pR'$.

For $\text{char}(R) = p^4$, Assume $s = h$, $t = 0$ and $\lambda = 0$ so that $R = R' \oplus \sum_{i=1}^s R'u_i$ with $pu_i = 0$, $0 \leq i \leq s$. The multiplication on R is defined by;

$$(a_o, \bar{a}_1, \dots, \bar{a}_h)(b_o, \bar{b}_1, \dots, \bar{b}_h) = (a_o b_o, a_o \bar{b}_1 + \bar{a}_1 b_o, \dots, a_o \bar{b}_h + \bar{a}_h b_o)$$

where $\bar{a}_i, \bar{b}_j \in R'/pR'$ and $1 \leq i, j \leq s$.

As established in [5, 12, 13], respectively for $\text{char}(R) = p, p^2, p^3$ and p^4 , R is turned by these multiplications into a commutative ring with identity $(1, 0, 0, \dots, 0)$.

4 The Wiener Index of $\Gamma(R)$ and its Invariants for the Classes of Completely Primary Finite Rings

In this section, we present some findings on the Wiener index $W(\Gamma(R))$, the average disorder number $A(\Gamma(R))$ and the average distance index $\mu(\Gamma(R))$ of the zero divisor graph $\Gamma(R)$. The Wiener index denoted as W and also known as path number or the Wiener number is a graph index defined on a graph with n nodes as

$$W = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n [d]_{ij}$$

where $[d]_{ij}$ is the graph distance matrix. The Wiener index $W(\Gamma(R))$ of the graph G with vertex count $|V(\Gamma(R))|$ has a relationship with the average disorder number of the zero divisor graph $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the average distance $\mu(\Gamma(R))$ between the vertices of $\Gamma(R)$ which is given by

$$\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{|V(\Gamma(R))|}{2}}.$$

In chemical graph theory, computations for Wiener indices for cyclic carbon-chained organic compounds and its applications cannot be underestimated. This index has been quite handy in determination of the boiling points and polarity number of alkanes and their branched isomers. Further, the most common natural field in the application of the Wiener index is the quantitative structure relationships especially in the estimation of emission spectra of the ultra violet radiations of α and β -unsaturated Ketone. We therefore present the following results on the Wiener index of $\Gamma(R)$ and other results describing average disorder number and the average distance indices of $\Gamma(R)$ due to their close interdependence with the Wiener index.

Proposition 1. Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Construction I. Then for any prime integer p and $r, s \in \mathbb{Z}^+$ with s fixed, the Wiener index, $W(\Gamma(R))$

$$= \begin{cases} \frac{1}{2}(2p^{\frac{(s^2+3s)}{2}-1}r + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{\frac{(s^2+3s)}{2}r} - 5p^{\frac{(s^2+3s)}{2}-1}r + 2), & \text{if } \text{char}(R) = p; \\ \frac{1}{2}(2p^{\frac{(s^2+3s)}{2}+2}r + p^{2(\frac{(s^2+3s)}{2})r} - p^{\frac{(s^2+3s)}{2}r} - 5p^{\frac{(s^2+3s)}{2}r} + 2), & \text{if } \text{char}(R) = p^2, pu_i = 0; \\ \frac{1}{2}(2p^{\frac{(2s^2+8s+4)}{2}r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{\frac{(s^2+5s+2)}{2}r} - 5p^{\frac{(s^2+3s)}{2}r} + 2), & \text{if } \text{char}(R) = p^2, pu_i \neq 0; \\ \frac{1}{2}(2p^{\frac{(2s^2+10s+6)}{2}r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 5p^{\frac{(s^2+5s+2)}{2}r} + 2), & \text{if } \text{char}(R) = p^3. \end{cases}$$

Proof. Case(i): $\text{Char}(R) = p$. The maximum degree of $v_i \in V(\Gamma(R)) = p^{\frac{(s^2+3s)}{2}r} - 2$ and there are $p^{\frac{(s^2+3s)}{2}-1}r - 1$ of such vertices. This is due to the fact that $|ann(Z(R))^*| = p^{\frac{(s^2+3s)}{2}-1}r - 1$. Therefore, the sum of minimum distances between v_i of a maximum degree and any other vertex $v_j \in V(\Gamma(R))$ is $(p^{\frac{(s^2+3s)}{2}r} - 2)(p^{\frac{(s^2+3s)}{2}-1}r - 1)$. For the vertices of minimum degrees, each is of degree $p^{\frac{(s^2+3s)}{2}-1}r - 1$ and sum of the distances between a vertex of minimum degree and any other vertex in the set of vertices of minimum degree is $p^{\frac{(s^2+3s)}{2}r} - 2$. If they are $p^{\frac{(s^2+3s)}{2}-1}r$ in number, then from the argument we obtain the sum as $((p^{\frac{(s^2+3s)}{2}-1}r - 1) + (p^{\frac{(s^2+3s)}{2}r} - 2))p^{\frac{(s^2+3s)}{2}-1}r$. The multiple $\frac{1}{2}$ handles the fact that each path between v_i and v_j is also counted as the path between v_j and v_i hence

$$\begin{aligned} W(\Gamma(R)) &= \frac{1}{2}(p^{\frac{(s^2+3s)}{2}r} - 2)(p^{\frac{(s^2+3s)}{2}-1}r - 1) + ((p^{\frac{(s^2+3s)}{2}-1}r - 1) + (p^{\frac{(s^2+3s)}{2}r} - 2))p^{\frac{(s^2+3s)}{2}-1}r = \\ &= \frac{1}{2}(p^{\frac{(s^2+3s)}{2}-1}r - p^{\frac{(s^2+3s)}{2}r} - 2p^{\frac{(s^2+3s)}{2}-1}r + 2 + p^{2(\frac{(s^2+3s)}{2}-1)r} + p^{\frac{(s^2+3s)}{2}-1}r - 3p^{\frac{(s^2+3s)}{2}-1}r) = \\ &= \frac{1}{2}((2p^{\frac{(s^2+3s)}{2}-1}r + p^{2(\frac{(s^2+3s)}{2}-1)r} - 5p^{\frac{(s^2+3s)}{2}-1}r - p^{\frac{(s^2+3s)}{2}r} + 2)). \end{aligned}$$

Case(ii): $\text{Char}(R) = p^2$ where $pu_i = 0$.

Using the same argument similar to $\text{char}(R) = p$, we obtain

$$\begin{aligned} W(\Gamma(R)) &= \frac{1}{2}((p^{\frac{(s^2+3s)}{2}+2}r - 2)(p^{\frac{(s^2+3s)}{2}r} - 1) + ((p^{\frac{(s^2+3s)}{2}r} - 1) + (p^{\frac{(s^2+3s)}{2}+2}r - 2))p^{\frac{(s^2+3s)}{2}r} = \\ &= \frac{1}{2}(p^{\frac{(s^2+3s)}{2}+2}r - p^{\frac{(s^2+3s)}{2}r} - 2p^{\frac{(s^2+3s)}{2}r} + 2 + p^{2(\frac{(s^2+3s)}{2}+2)r} + p^{\frac{(s^2+3s)}{2}+2}r - 3p^{\frac{(s^2+3s)}{2}r} = \\ &= \frac{1}{2}(2p^{\frac{(s^2+3s)}{2}+2}r + p^{2(\frac{(s^2+3s)}{2}+2)r} - p^{\frac{(s^2+3s)}{2}r} - 5p^{\frac{(s^2+3s)}{2}r} + 2). \end{aligned}$$

Case(iii): $\text{Char}(R) = p^2$ where $pu_i \neq 0$,

$$W(\Gamma(R)) = \frac{1}{2}((p^{\frac{(s^2+5s+2)}{2}r} - 2)(p^{\frac{(s^2+3s)}{2}r} - 1) + ((p^{\frac{(s^2+3s)}{2}r} - 1) + (p^{\frac{(s^2+5s+2)}{2}r} - 2))p^{\frac{(s^2+3s)}{2}r} =$$

$$\begin{aligned} & \frac{1}{2}(p^{(\frac{(2s^2+8s+4)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 2p^{(\frac{(s^2+3s+2)}{2})r} + 2 + (p^{2(\frac{(s^2+3s+2)}{2})r} + p^{(\frac{(2s^2+8s+4)}{2})r} - 3p^{(\frac{(s^2+3s+2)}{2})r}) = \\ & \frac{1}{2}(2p^{(\frac{(s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2). \end{aligned}$$

Case(iv): $\text{Char}(R) = p^3$.

We have

$$\begin{aligned} W(\Gamma(R)) &= \frac{1}{2}((p^{(\frac{(s^2+5s+4)}{2})r} - 2)(p^{(\frac{(2s^2+5s+2)}{2})r} - 1) + ((p^{(\frac{(s^2+5s+2)}{2})r} - 1) + (p^{(\frac{(s^2+5s+4)}{2})r} - 2)p^{(\frac{(s^2+5s+2)}{2})r}) = \\ & \frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} - p^{(\frac{(s^2+5s+4)}{2})r} - 2p^{(\frac{(s^2+5s+2)}{2})r} + 2 + (p^{(\frac{(2s^2+10s+4)}{2})r} + p^{(\frac{(2s^2+10s+6)}{2})r} - 3p^{(\frac{(s^2+5s+2)}{2})r}) = \\ & \frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{(\frac{(2s^2+10s+4)}{2})r} - p^{(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2). \end{aligned}$$

□

Proposition 2. Let $\Gamma(R)$ be the zero divisor graph of the classes of rings given by Construction I and $W(\Gamma(R))$ be its Wiener index. Then for any prime integer p , positive integers r, s with s fixed, the average distance of $\Gamma(R)$

$$\mu(\Gamma(R)) = \begin{cases} \frac{\frac{1}{2}(2p^{(\frac{(2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 5p^{(\frac{(s^2+3s)}{2}-1)r} + 2)}{(p^{(\frac{(s^2+3s)}{2})r} - 1)(p^{(\frac{(s^2+3s)}{2})r} - 2)}}, & \text{Char}(R) = p; \\ \frac{\frac{1}{2}(2p^{(\frac{(2(s^2+3s)}{2}+2)r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 5p^{(\frac{(s^2+3s)}{2})r} + 2)}{(p^{(\frac{(s^2+3s+2)}{2})r} - 1)(p^{(\frac{(s^2+3s+2)}{2})r} - 2)}}, & \text{Char}(R) = p^2, pu_i = 0; \\ \frac{\frac{1}{2}(2p^{(\frac{(2s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2)}{(p^{(\frac{(s^2+5s+2)}{2})r} - 1)(p^{(\frac{(s^2+5s+2)}{2})r} - 2)}}, & \text{Char}(R) = p^2, pu_i \neq 0; \\ \frac{\frac{1}{2}(2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2)}{(p^{(\frac{(s^2+5s+4)}{2})r} - 1)(p^{(\frac{(s^2+5s+4)}{2})r} - 2)}}, & \text{Char}(R) = p^3. \end{cases}$$

Proof. The Proof follows from the previous Proposition 1 and the fact that $\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{|V(\Gamma(R))|}{2}}$. □

Proposition 3. Let $\Gamma(R)$ be the zero divisor graph of the classes of rings described by Construction I. Then for any prime integer p , positive integers r, s and s fixed, the average disorder number of the zero divisor graph

$$A(\Gamma(R)) = \begin{cases} \frac{2p^{(\frac{(2(s^2+3s)}{2}-1)r} + p^{2(\frac{(s^2+3s)}{2}-1)r} - p^{(\frac{(s^2+3s)}{2})r} - 5p^{(\frac{(s^2+3s)}{2}-1)r} + 2)}{(p^{(\frac{(s^2+3s)}{2})r} - 1)}, & \text{Char}(R) = p; \\ \frac{2p^{(\frac{(2(s^2+3s)}{2}+2)r} + p^{2(\frac{(s^2+3s)}{2})r} - p^{(\frac{(s^2+3s+2)}{2})r} - 5p^{(\frac{(s^2+3s)}{2})r} + 2)}{(p^{(\frac{(s^2+3s+2)}{2})r} - 1)}, & \text{Char}(R) = p^2, pu_i = 0; \\ \frac{2p^{(\frac{(2s^2+8s+4)}{2})r} + p^{2(\frac{(s^2+3s+2)}{2})r} - p^{(\frac{(s^2+5s+2)}{2})r} - 5p^{(\frac{(s^2+3s+2)}{2})r} + 2}{(p^{(\frac{(s^2+5s+2)}{2})r} - 1)}, & \text{Char}(R) = p^2, pu_i \neq 0; \\ \frac{2p^{(\frac{(2s^2+10s+6)}{2})r} + p^{2(\frac{(2s^2+10s+4)}{2})r} - p^{2(\frac{(s^2+5s+4)}{2})r} - 5p^{(\frac{(s^2+5s+2)}{2})r} + 2}{(p^{(\frac{(s^2+5s+4)}{2})r} - 1)}, & \text{Char}(R) = p^3. \end{cases}$$

Proof. Using the ratio $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the Wiener indices obtained in Proposition 1, the result follows. □

Proposition 4. Let $\Gamma(R)$ be the zero divisor graph of a ring of characteristic p given by Construction II. Then the Wiener index,

$$W(\Gamma(R)) = \frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2).$$

Proof. Let V_1, V_2 and V_3 be the order of partitioning of the vertices in $\Gamma(R)$. Consider $V_1 = Ann(Z(R)) \setminus \{0\}$. we have that $|V_1| = p^r - 1$ and the degree of every vertex $x \in V_1$, $deg(x \in V_1) = p^{hr} - 1 - 1 = p^{hr} - 2$ due to avoidance of self annihilation. Therefore the sum of the distances between vertices $x \in V_1$ and any other vertex in $V(\Gamma(R))$ is $(p^{hr} - 2)(p^r - 1) = p^{(h+1)r} - p^{hr} - 2p^r + 2$.

Next, consider the set of vertices V_2 which are linked by edges with vertices in V_1 and among themselves such that $d(x, y \in V_2) = p^{hr} + 2$. Since $|V_2| = p^r$, we have that

$$\sum_{y \in V_2} d(x, y) = p^r(p^{hr} + 2) = p^{(h+1)r} + 2p^r.$$

Finally, let V_3 be the set of vertices such that $z \in V_3$ is only adjacent $x \in V_1$. Therefore, $|V_3| = (p^{hr} - 1) - (p^r + p^r - 1) = p^{hr} - 2p^r$ and the distance between every vertex in V_3 and all other vertices in $\Gamma(R)$ is $p^{hr} + p^r + 1$. Therefore, the sum of the minimum distances between $z \in V_3$ and vertices $V(\Gamma(R))$ is obtained as

$$\sum_{z \in V_3} d(x, z) = (p^{hr} - 2p^r)(p^{hr} + p^r + 1) =$$

$$p^{2hr} + p^{(h+1)r} + p^{hr} - 2p^{(h+1)r} - 2p^{2r} - 2p^r = p^{2hr} - p^{(h+1)r} + p^{hr} - 2p^{2r} - 2p^r.$$

Therefore,

$$W(\Gamma(R)) = \frac{1}{2}(p^{(h+1)r} - p^{hr} - 2p^r + 2 + p^{(h+1)r} + 2p^r + p^{2hr} - p^{(h+1)r} + p^{hr} - 2p^{2r} - 2p^r)$$

which simplifies to

$$\frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2p^{2r} - 2p^r + 2) = \frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2).$$

□

Proposition 5. Let $W(\Gamma(R))$ be the Wiener index of the zero divisor graph of a ring of characteristic p given by Construction II. Then the average disorder number $A(\Gamma(R))$ and the average distance index $\mu(\Gamma(R))$ are given by;

$$(i) A(\Gamma(R)) = \frac{p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2}{p^{hr} - 1}.$$

$$(ii) \mu(\Gamma(R)) = \frac{\frac{1}{2}(p^{(h+1)r} + p^{2hr} - 2(p^{2r} + p^r) + 2)}{(p^{hr} - 1)(p^{hr} - 2)}.$$

Proof. From the fact that $|V(\Gamma(R))| = p^{hr} - 1$, using the ratios $A(\Gamma(R)) = \frac{2W(\Gamma(R))}{|V(\Gamma(R))|}$ and the average distance $\mu(\Gamma(R))$ between the vertices of $\Gamma(R)$, $\mu(\Gamma(R)) = \frac{W(\Gamma(R))}{\binom{|V(\Gamma(R))|}{2}}$, the results follows from the

previous Proposition 4. □

Proposition 6. Let $\Gamma(R)$ be the zero divisor graph of classes of rings given by the Construction II of characteristics p^2, p^3 and p^4 and $W(\Gamma(R))$ be the Wiener index. Then,

$$W(\Gamma(R)) \begin{cases} \frac{1}{2}(p^{(h+2)r} + p^{(2h+1)r} - 2(p^{3r} + p^{2r})), & \text{when } Char(R) = p^2; \\ \frac{1}{2}(p^{(h+3)r} + p^{(2h+2)r} - 2(p^{4r} + p^{3r})), & \text{when } Char(R) = p^3; \\ \frac{1}{2}(p^{(h+4)r} + p^{(2h+3)r} - 2(p^{5r} + p^{4r})), & \text{when } Char(R) = p^4. \end{cases}$$

Proof. The proof follows from Proposition 4. □

5 Binding Numbers of the Classes of Completely Primary Finite Commutative Rings

The reason for rapid and steady development in the study of the binding number $b(\Gamma(R))$ in graphs is related to its diversity in both theoretical and real world applications. This is an important characteristic quantity which is applicable in understanding the graph characteristics and vulnerability. We provide an investigation of the binding numbers of some classes of cube radical zero and power four radical zero completely primary finite rings in the following results.

Proposition 7. *Let $\Gamma(R)$ be the zero divisor graph of a ring of characteristic p given by Construction I. Then the binding number,*

$$b(\Gamma(R)) = \frac{p^{(\frac{s^2+3s}{2}-1)r} - 1}{p^{(\frac{s^2+3s}{2})r} - p^{(\frac{s^2+3s}{2}-1)r}}.$$

Proof. Due to the fact that $N(S) = \text{Ann}(Z(R)) \setminus \{0\}$ and $S = V(\Gamma(R)) \setminus N(S)$, we have $|N(S)| = p^{(\frac{s^2+3s}{2}-1)r} - 1$.

But $|S| = |V(\Gamma(R)) \setminus N(S)| = p^{(\frac{s^2+3s}{2})r} - 1 - (p^{(\frac{s^2+3s}{2}-1)r} - 1) = p^{(\frac{s^2+3s}{2})r} - 1 - p^{(\frac{s^2+3s}{2}-1)r} + 1 = p^{(\frac{s^2+3s}{2})r} - p^{(\frac{s^2+3s}{2}-1)r}$. Therefore, the binding number of $\Gamma(R)$,

$$b(\Gamma(R)) = \frac{|N(S)|}{|S|} = \frac{p^{(\frac{s^2+3s}{2}-1)r} - 1}{p^{(\frac{s^2+3s}{2})r} - p^{(\frac{s^2+3s}{2}-1)r}}.$$

□

Proposition 8. *Let $\Gamma(R)$ be the zero divisor graph of a ring of characteristic p^2 given by Construction I. Then the binding number,*

$$b(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{s^2+3s}{2})r} - 1}{p^{(\frac{s^2+3s+2}{2})r} - p^{(\frac{s^2+3s}{2})r}}, & pu_i = 0; \\ \frac{p^{(\frac{s^2+3s+2}{2})r} - 1}{p^{(\frac{s^2+5s+2}{2})r} - p^{(\frac{s^2+3s+2}{2})r}}, & pu_i \neq 0. \end{cases}$$

Proof. Case(i), $pu_i = 0$.

Since $N(S) = \text{Ann}(Z(R)) \setminus \{0\}$ and $S = V(\Gamma(R)) \setminus N(S)$,

it is established that $|N(S)| = p^{(\frac{s^2+3s}{2})r} - 1$.

But $|S| = |V(\Gamma(R)) \setminus N(S)| = p^{(\frac{s^2+3s+2}{2})r} - 1 - (p^{(\frac{s^2+3s}{2})r} - 1) = p^{(\frac{s^2+3s+2}{2})r} - 1 - p^{(\frac{s^2+3s}{2})r} + 1 = p^{(\frac{s^2+3s+2}{2})r} - p^{(\frac{s^2+3s}{2})r}$. Therefore, the binding number of $\Gamma(R)$,

$$b(\Gamma(R)) = \frac{|N(S)|}{|S|} = \frac{p^{(\frac{s^2+3s}{2})r} - 1}{p^{(\frac{s^2+3s+2}{2})r} - p^{(\frac{s^2+3s}{2})r}}.$$

Case(ii), $pu_i \neq 0$.

Given that $|V(\Gamma(R))| = p^{(\frac{s^2+5s+2}{2})r} - 1$, consider the $\text{Ann}(Z(R))^* = N(S)$. We notice that $N(S) = p^{(\frac{s^2+3s+2}{2})r} - 1$ and therefore

$$\begin{aligned} |S| &= (p^{(\frac{s^2+5s+2}{2})r} - 1) - (p^{(\frac{s^2+3s+2}{2})r} - 1) = p^{(\frac{s^2+5s+2}{2})r} - 1 - p^{(\frac{s^2+3s+2}{2})r} + 1 = \\ &= p^{(\frac{s^2+5s+2}{2})r} - p^{(\frac{s^2+3s+2}{2})r}. \end{aligned}$$

Clearly,

$$b(\Gamma(R)) = \frac{p^{(\frac{s^2+3s+2}{2})r} - 1}{p^{(\frac{s^2+5s+2}{2})r} - p^{(\frac{s^2+3s+2}{2})r}}.$$

□

Proposition 9. Let R be a ring of characteristic p^3 described by Construction I and $\Gamma(R)$ be its zero divisor graph. Then,

$$b(\Gamma(R)) = \frac{p^{(\frac{s^2+3s+2}{2})r} - 1}{p^{(\frac{s^2+5s+4}{2})r} - (p^{(\frac{s^2+3s+2}{2})r})}.$$

Proof. The proof follows from the fact that

$$|N(S)| = |Ann(Z(R))^*| = p^{(\frac{s^2+3s+2}{2})r} - 1$$

for the rings of characteristic p^3 and $S = V(\Gamma(R)) \setminus N(S)$. We have that $|S| = (p^{(\frac{s^2+5s+4}{2})r} - 1) - (p^{(\frac{s^2+3s+2}{2})r} - 1) = p^{(\frac{s^2+5s+4}{2})r} - 1 - p^{(\frac{s^2+3s+2}{2})r} + 1 = p^{(\frac{s^2+5s+4}{2})r} - p^{(\frac{s^2+3s+2}{2})r}$

Using the ratio $\frac{|N(S)|}{|S|}$, we obtain

$$b(\Gamma(R)) = \frac{p^{(\frac{s^2+3s+2}{2})r} - 1}{p^{(\frac{s^2+5s+4}{2})r} - p^{(\frac{s^2+3s+2}{2})r}}.$$

□

Proposition 10. Let R be rings given by Constructions II and $\Gamma(R)$ be the zero divisor graph. Then the binding number,

$$b(\Gamma(R)) = \begin{cases} \frac{1}{p^{hr}-2}, & \text{when } Char(R) = p; \\ \frac{p^{hr}-1}{p^{(h+3)r}-p^{hr}}, & \text{when } Char(R) = p^2; \\ \frac{p^{(h-1)r}}{p^{(h+3)r}-p^{(h-1)r}}, & \text{when } Char(R) = p^3 \text{ and } p^4. \end{cases}$$

Proof. With the fact that $N(S) = Ann(Z(R))^*$ and $S = V(\Gamma(R)) \setminus N(S)$, applying the ratio $\frac{|N(S)|}{|S|}$, the results follows from the Proofs in Propositions 7, 8 and 9. □

6 Bounds On the Zagreb Indices of the Zero Divisor Graphs $\Gamma(R)$ of the Classes of Completely Primary Finite Rings

Let $G = \Gamma(R)$ be a simple graph such that $G = (V, E)$ whose vertex set $V(G)$ consist of elements $\{v_1, \dots, v_n\}$ such that $|V(G)| = n$ and the set of edges $E(G)$ of order m . Given that the minimum degree of G is denoted by $\delta(G)$ and $\Delta(G)$ is the maximum degree. Let $d_i = \deg_{\Gamma(R)}(v_i)$, $i = 1, 2, \dots, n$ be the vertex degrees of $v_i \in \Gamma(R)$ so that $d_1 \geq d_2 \geq \dots \geq d_n$. The first Zagreb index is the sum of the squares of degrees of the vertices and the second Zagreb index is the sum of the products of the degrees of the pairs of adjacent vertices. We denote the first and second Zagreb indices of $\Gamma(R)$ by $Z_1(\Gamma(R))$ and $Z_2(\Gamma(R))$ respectively. Therefore,

$$Z_1(\Gamma(R)) = \sum_{i=1}^n (\deg(v_i))^2,$$

$$Z_2(\Gamma(R)) = \sum_{i,j=1}^n (\deg(v_i))(\deg(v_j))$$

where v_i, v_j are adjacent.

6.1 Bounds On First Zagreb Index, $Z_1(\Gamma(R))$ of the Classes of Completely Primary Finite Rings

We make use of Lemma 1, 2 and 3 to generally describe some of the results on the bounds on the first Zagreb index of $\Gamma(R)$ of the classes of power four radical zero rings.

Proposition 11. *Let R be classes of rings described in Construction II and $\Gamma(R)$ be the zero divisor graph with m edges such that $|\Gamma(R)| = p^{(h+(k-1))r} - 1$. If $\Delta(\Gamma(R))$ and $\delta(\Gamma(R))$ are the maximum and minimum degrees of $\Gamma(R)$ respectively, then for any $r, k \in \mathbb{Z}^+$, p prime, and h is the dimension of R' -module U ,*

$$(i) Z_1(\Gamma(R)) \geq \frac{((\Delta(\Gamma(R)))^2 + (2m - \Delta(\Gamma(R))))^2}{(p^{(h+(k-1))r} - 2)} + \frac{2(p^{(h+(k-1))r} - 3)}{(p^{(h+(k-1))r} - 2)^2} \cdot (\Delta_2(\Gamma(R)) - \delta(\Gamma(R)))^2, \text{ where } \Delta_2(\Gamma(R)) \text{ is the second maximum degree of } \Gamma(R).$$

$$(ii) Z_1(\Gamma(R)) \leq 4m^2 + 2((\Delta(\Gamma(R)))^2 - 4m((\Delta(\Gamma(R)))) - ((p^{(h+(k-1))r} - 2)((p^{(h+(k-1))r} - 3)) \left[\frac{T(\Gamma(R))}{(p^{(h+(k-1))r} - 2)\Delta(\Gamma(R))} (I(\Gamma(R))) - \frac{1}{\Delta(\Gamma(R))} \right] \frac{2}{p^{(h+(k-1))r} - 3}.$$

Proof. (i) From Lemma 3, set $r = p^{(h+(k-1))r} - 2$, $a_i = d_{i+1}$, $b_i = 1$, $i = 1, 2, \dots, r$ which results to $(p^{(h+(k-1))r} - 2)(\sum_{i=2}^{p^{(h+(k-1))r}-1} d_i^2) - (\sum_{j=2}^{p^{(h+(k-1))r}-2} d_j^2)^2 = \sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)^2$. This results to $(p^{(h+(k-1))r})(Z_1(\Gamma(R)) - (\Delta(\Gamma(R)))^2) = (2m - (\Delta(\Gamma(R)))^2 + \sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j))$. Now, $\sum_{i,j=2}^{p^{(h+(k-1))r}-1} |d_i - d_j| = (p^{(h+(k-1))r} - 3)d_2 - \sum_{i=3}^{p^{(h+(k-1))r}-1} d_i + \sum_{i,j=3}^{p^{(h+(k-1))r}-2} |d_i - d_j| + \sum_{i=3}^{p^{(h+(k-1))r}-4} d_i - (p^{(h+(k-1))r} - 4)d_{p^{(h+(k-1))r}-1} = (p^{(h+(k-1))r} - 3)(d_2 - d_{p^{(h+(k-1))r}-1}) + \sum_{i,j=3}^{p^{(h+(k-1))r}-2} |d_i - d_j| \geq (p^{(h+(k-1))r} - 3)(\Delta_2(\Gamma(R)) - (\delta(\Gamma(R))))(***).$ which results to

$$\frac{\sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)^2}{(p^{(h+(k-1))r} - 2)(p^{(h+(k-1))r} - 3)} \geq \frac{\sum_{i,j=2}^{p^{(h+(k-1))r}-1} |d_i - d_j|}{(p^{(h+(k-1))r} - 2)(p^{(h+(k-1))r} - 3)}.$$

From this we obtain

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)^2 \geq \frac{2}{(p^{(h+(k-1))r} - 2)(p^{(h+(k-1))r} - 3)} \left(\sum_{i,j=2}^{p^{(h+(k-1))r}-1} |d_i - d_j|^2 \right)$$

which together with (***) gives

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)^2 \geq \frac{2(p^{(h+(k-1))r}-3)}{p^{(h+(k-1))r}-2} (\Delta_2(\Gamma(R)) - \delta(\Gamma(R)))^2 \text{ which is simplified to give the result as desired.}$$

(ii) We let the simple topological index of $\Gamma(R)$ be $T(\Gamma(R)) = \prod_{i=1}^{p^{(h+(k-1))r}-1} d_i$ and the inverse degree of $\Gamma(R)$ be $I(G) = \sum_{i=1}^{p^{(h+(k-1))r}-1} \frac{1}{d_i}$.

Set $r = p^{(h+(k-1))r} - 2$, $a_i = d_{i+1}$, $i = 1, 2, \dots, r$. Making use of Lemma 3, we have

$$\begin{aligned} & \sum_{i,j=2}^{p^{(h+(k-1))r}-1} d_i d_j \geq \\ & \frac{(p^{(h+(k-1))r} - 2)(p^{(h+(k-1))r} - 3)}{2} \left[\frac{1}{p^{(h+(k-1))r} - 2} \prod_{j=2}^{p^{(h+(k-1))r}-2} d_j \sum_{i=2}^{p^{(h+(k-1))r}-1} \frac{1}{d_i} \right]^{\frac{2}{(p^{(h+(k-1))r}-3)}} = \\ & \frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2} \left[\frac{1}{(p^{(h+(k-1))r}-2)(\Delta(\Gamma(R)))} \prod_{j=2}^{p^{(h+(k-1))r}-2} d_j \left(\sum_{j=2}^{p^{(h+(k-1))r}-2} \frac{1}{d_j} - \frac{1}{\Delta(\Gamma(R))} \right) \right]^{\frac{2}{(p^{(h+(k-1))r}-3)}} = \\ & \frac{(p^{(h+(k-1))r}-2)(p^{(h+(k-1))r}-3)}{2} \left[\frac{T(\Gamma(R))}{(p^{(h+(k-1))r}-2)(\Delta(\Gamma(R)))} (I(\Gamma(R)) - \frac{1}{\Delta(\Gamma(R))}) \right]^{\frac{2}{(p^{(h+(k-1))r}-3)}}. \end{aligned}$$

From this we have that

$$\sum_{i,j=2}^{p^{(h+(k-1))r}-1} (d_i - d_j)^2 =$$

$$p^{(h+(k-1))r} - 3 \sum_{i=2}^{p^{(h+(k-1))r}-1} d_i^2 - 2 \sum_{i,j=2}^{p^{(h+(k-1))r}-1} d_i d_j \leq$$

$$(p^{(h+(k-1))r} - 3)(Z_1(\Gamma(R)) - \Delta(\Gamma(R))(p^{(h+(k-1))r} - 2)(p^{(h+(k-1))r} - 3) \times$$

$$\left[\frac{T(\Gamma(R))}{(p^{(h+(k-1))r} - 2)\Delta(\Gamma(R))} (I(\Gamma(R))) - \frac{1}{\Delta(\Gamma(R))} \right]^{\frac{2}{p^{(h+(k-1))r} - 3}}.$$

Which describes the upper bound on $Z_1(\Gamma(R))$. $\square$

Proposition 12. Let R be classes of rings described in Construction II and $\Gamma(R)$ be the zero divisor graph with m edges such that $|\Gamma(R)| = p^{(h+(k-1))r} - 1$. If $\Delta(\Gamma(R))$ is the maximum degree of each $v_i \in \Gamma(R)$ then,

$$Z_1(\Gamma(R)) \leq (p^{(h+(k-1))r} m - \Delta(\Gamma(R))((p^{(h+(k-1))r} - 1) - \Delta(\Gamma(R)))) + \frac{2(m - \Delta(\Gamma(R)))}{p^{(h+(k-1))r} - 3}$$

Proof. Let v_1 be a vertex of maximum degree in $\Gamma(R)$ and $H = \{v_{i1}, v_{i2}, \dots, v_{i\Delta(\Gamma(R))}\} \subseteq V(\Gamma(R))$ be the vertex set consisting of vertices adjacent to v_1 . Let $\Gamma_{v_1}(R) = \Gamma(R) - v_1$ be the induced subgraph of $\Gamma(R)$ obtained by removing a vertex of minimum degree and d'_i be its degree. Then we have

$$d'_i = \begin{cases} d_{i-1}, & v_i \in H; \\ d_i, & v_i \in V(\Gamma(R)) \setminus H \end{cases}$$

Since $\Gamma_{v_1}(R)$ has $p^{(h+(k-1))r} - 2$ vertices and $m - \Delta(\Gamma(R))$ the maximum degree of a vertex in the subgraph, then by Lemma 1 we obtain

$$\sum_{v_i \in V(\Gamma(R))} d_i = 2(m - \Delta(\Gamma(R))),$$

Therefore

$$\sum_{v_i \in V(\Gamma(R))} 2(m - \Delta(\Gamma(R))) \leq (m - \Delta(\Gamma(R))) \left[\frac{2((m - \Delta(\Gamma(R))))}{p^{(h+(k-1))r} - 3} + (p^{(h+(k-1))r} + 2) \right].$$

Now,

$$Z_1(\Gamma(R)) = (\Delta(\Gamma(R)))^2 + \sum_{v_i \in H} d_i^2 + \sum_{v_i \in V(\Gamma(R)) \setminus H, i \neq 1} d_i^2 =$$

$$(\Delta(\Gamma(R)))^2 + \sum_{v_i \in H} (d'_i + 1)^2 + \sum_{v_i \in V(\Gamma_{v_1}(R)) \setminus H} d_i'^2 =$$

$$(\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 2 \sum_{v_i \in H} d'_i + \sum_{v_i \in V(\Gamma_{v_1}(R))} d_i'^2, \quad |H| = \Delta(\Gamma(R))$$

$$\leq (\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 2 \sum_{v_i \in V(\Gamma_{v_1}(R))} d'_i + \sum_{v_i \in V(\Gamma_{v_1}(R))} d_i'^2$$

$$\leq (\Delta(\Gamma(R)))^2 + \Delta(\Gamma(R)) + 4(m - \Delta(\Gamma(R))) + (m - \Delta(\Gamma(R))) \left[\frac{2((m - \Delta(\Gamma(R))))}{p^{(h+(k-1))r} + 2} \right].$$

$\square$

6.2 Bounds On the Second Zagreb Index, $Z_2(\Gamma(R))$ of the Classes of Completely Primary Finite Rings

Using the previous results obtained in Propositions 11 and 12, we generally present some results on the upper and lower bounds on the second Zagreb index $Z_2(\Gamma(R))$ from the maximum and minimum degrees of $\Gamma(R)$ with m -edges as follows.

Proposition 13. *Let R be classes of rings described in Construction II and $\Gamma(R)$ is the zero divisor graph with m edges such that $|V(\Gamma(R))| = p^{(h+(k-1))r} - 1$ and that $\Delta(\Gamma(R))$ and $\delta(\Gamma(R))$ are the maximum and minimum degrees of $\Gamma(R)$ respectively. Then for any prime integer p , positive integers r, k , and h the dimension R' -module U ,*

$$(i) \quad Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(h+(k-1))r} - 2)\Delta(\Gamma(R)) + \frac{1}{2}(\Delta(\Gamma(R)) - 2)[(\Delta(\Gamma(R)))^2 + \frac{(2m - \Delta(\Gamma(R)))^2}{p^{(h+(k-1))r} - 2} + \frac{2(p^{(h+(k-1))r} - 3)}{(p^{(h+(k-1))r} - 2)^2}(\Delta(\Gamma(R)) - \delta(\Gamma(R)))^2].$$

$$(ii) \quad Z_2(\Gamma(R)) \geq 2m^2 - m(p^{(h+(k-1))r} - 2)\delta(\Gamma(R)) + \frac{1}{2}(\delta(\Gamma(R)) - 1)[m(p^{(h+(k-1))r} - \Delta(\Gamma(R)))(p^{(h+(k-1))r} - \Delta(\Gamma(R))) + \frac{2(m - \Delta(\Gamma(R)))^2}{p^{(h+(k-1))r} - 3}].$$

Proof. (i) Given that μ is the average distance of the vertices adjacent to $v_i \in V(\Gamma(R))$, we have that

$$Z_2(\Gamma(R)) = \frac{1}{2} \sum_{i=1}^{p^{(h+(k-1))r} - 1} d_i^2 \mu.$$

We have that

$$\frac{1}{2} \sum_{i=1}^{p^{(h+(k-1))r} - 1} d_i [2m - d_i - ((p^{(h+(k-1))r} - 1) - d_i - 1)\Delta(\Gamma(R))] \leq$$

$$Z_2(\Gamma(R)) \leq \frac{1}{2} \sum_{i=1}^{p^{(h+(k-1))r} - 1} d_i [2m - d_i - ((p^{(h+(k-1))r} - 1) - d_i - 1)\delta(\Gamma(R))]$$

where $2m^2 - (p^{(h+(k-1))r} - 2)m\Delta(\Gamma(R)) + \frac{1}{2}(\Delta(\Gamma(R)) - 1)Z_1(\Gamma(R)) \leq$

$$Z_2(\Gamma(R)) \leq 2m^2 - (p^{(h+(k-1))r} - 2)m\delta(\Gamma(R)) + \frac{1}{2}(\delta(\Gamma(R)) - 1)Z_1(\Gamma(R)).$$

The inequality on the right hand side hold if and only if for every v_i , $d_i = p^{(h+(k-1))r} - 2$ or $d_j = \delta(\Gamma(R))$ for every v_i if v_i is non adjacent to v_j in $\Gamma(R)$.

Proof for (ii) follows from Proposition 12. □

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