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Research Article

Classification of Unit Groups of Five Radical Zero Completely Primary Finite Rings Whose First and Second Galois Ring Module Generators Are of the Order p^k, $k = 2, 3, 4$

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Let $R_0 = GR(p^{kr}, p^k)$ be a Galois maximal subring of R so that $R = R_0 \oplus U \oplus V \oplus W \oplus Y$, where U, V, W, and Y are R_0/pR_0 spaces considered as R_0-modules, generated by the sets $\{u_1, \cdots, u_e\}$, $\{v_1, \cdots, v_f\}$, $\{w_1, \cdots, w_g\}$, and $\{y_1, \cdots, y_h\}$, respectively. Then, R is a completely primary finite ring with a Jacobson radical $Z(R)$ such that $(Z(R))^5 = (0)$ and $(Z(R))^4 \neq (0)$. The residue field $R/Z(R)$ is a finite field $GF(p^r)$ for some prime p and positive integer r. The characteristic of R is p^k, where k is an integer such that $1 \leq k \leq 5$. In this paper, we study the structures of the unit groups of a commutative completely primary finite ring R with $p^{\psi}u_i = 0$, $\psi = 2, 3, 4$; $p^{\zeta}v_j = 0$, $\zeta = 2, 3$; $pw_k = 0$, and $py_l = 0$; $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, and $1 \leq l \leq h$.

1. Introduction

Completely primary finite rings with identity $1 \neq 0$ have been an active area of research in the recent years probably because they play a crucial role in the classification of finite rings. The unit groups of completely primary finite rings with maximal ideal $Z(R)$ such that $(Z(R))^3 = (0)$ with $(Z(R))^2 \neq (0)$ have been classified by Chikunji [1–3]. Oduor and Onyango [4] constructed a class of completely primary finite rings in which $(Z(R))^4 = (0)$ with $(Z(R))^3 \neq (0)$ and determined the structures of their group of units for all the characteristics of the ring R. Recently, Were et al. [5] gave a construction of a completely primary finite ring satisfying the conditions $(Z(R))^5 = (0)$; $(Z(R))^4 \neq (0)$ and further determined the unit groups restricted to some conditions. This construction involved idealization of R_0-modules, whose choice was based on Wilson [6]. Were et al. [7] classified the group of units of five radical zero completely primary finite rings with variant orders of second Galois ring module generators. The structure of unit groups for an abelian group have been determined by various techniques. Ayoub [8] studied groups which possess a particular type of series called j- diagram and determined the structure of the subgroups. In [9], Ayoub obtained various results based on the ideas regarding j-diagrams for the abelian p-groups. The j-diagram technique has been used by Alabiad and Alkhamees [10] to determine the structure of unit groups of a commutative chain ring where $(p-1)|k$ by fixing Ayoub's approach and introduced a system of generators for the unit groups as well as enumerating the generators. In what follows, R shall denote a finite completely primary ring, $Z(R)$ its maximal set of zero divisors (including zero). We shall also denote the coefficient Galois subring $GR(p^{kr}, p^k)$ of characteristic p^k and order p^{kr} of the ring R by R_0. For the previous related work, we refer to [2, 3, 5, 10, 11–13].

The remaining part of this paper is organized as follows. In Section 2, we give the construction of five radical zero completely primary finite rings whose structure of unit groups are considered in this paper. In Section 3, we investigate and determine the structure of the group of units R^* of R for all characteristics p^k, $1 \leq k \leq 5$, under the restrictions $p^{\psi}u_i = 0$, $p^{\zeta}v_j = 0$, $pw_k = 0$, $py_l = 0$, where $\psi = 2, 3, 4$; $\xi = 2, 3$;

$1 \le i \le e, 1 \le j \le f$, $1 \le k \le g$, and $1 \le l \le h$. Finally, in Section 4, we give the conclusion of this research and some areas which future researchers may dwell on. This is a continuation of the author's earlier research on the classification of unit groups of five radical zero commutative completely primary finite rings.

2. Preliminaries

The following result due to Wilson [6] formed the basis for the choice of the ring in this work as an R_0-module.

Let R be a completely primary finite ring of characteristic p^k with radical $Z(R)$ such that $R/Z(R) \cong GF(p^r)$. Then, there exists an independent generating set $b_1, b_2, \cdots, b_m$ of R as a left $GR(p^{kr}, p^r)$-module such that

(1) $b_1 = 1; b_2, \cdots, b_m \in Z(R)$

(2) $GR(p^{kr}, p^r)b_i$ is $(GR(p^{kr}, p^r), GR(p^{kr}, p^r))$-submodule of R

This paper considers a specific case of the general construction of five radical zero commutative completely primary finite rings where the automorphism is the identity in R_0 and the ring R is commutative.

2.1. Construction. Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k where p is a prime integer, $1 \le k \le 5$ and $r \in \mathbb{Z}^+$. Suppose U,V,W, and Y are R_0/pR_0-spaces considered R_0 modules generated by e, f, g, and h elements, respectively, such that the corresponding generating sets are $\{u_1, \cdots, u_e\}$, $\{v_1, \cdots, v_f\}$, $\{w_1, \cdots, w_g\}$, and $\{y_1, \cdots, y_h\}$, so that $R = R_0 \oplus U \oplus V \oplus W \oplus Y$ is an additive abelian group. Then, on the additive group, we define multiplication by the following relations:

(i) If $k = 1$, then $u_iu_{i'} = u_{i'}u_i = v_j$, $u_iv_j = v_ju_i = w_k$, $u_iw_k = w_ku_i = y_l$, $u_iy_l = y_lu_i = 0$, $v_jv_{j'} = v_{j'}v_j = y_l$, $v_jw_k = w_kv_j = 0$, $v_jy_l = y_lv_j = 0$, $w_kw_{k'} = w_{k'}w_k = 0$, $w_ky_l = y_lw_k = 0$, $y_ly_{l'} = y_{l'}y_l = 0$

(ii) If $k = 2$, then $u_iu_{i'} = u_{i'}u_i = pr_0 + pu_i + v_j$, $u_iv_j = v_ju_i = pu_i + w_k$, $u_iw_k = w_ku_i = pu_i + y_l$, $u_iy_l = y_lu_i = pu_i$, $v_jv_{j'} = v_{j'}v_j = y_l$, $v_jw_k = w_kv_j = 0$, $v_jy_l = y_lv_j = 0$, $w_kw_{k'} = w_{k'}w_k = 0$, $w_ky_l = y_lw_k = 0$, $y_ly_{l'} = y_{l'}y_l = 0$

(iii) If $3 \le k \le 5$, then $u_iu_{i'} = u_{i'}u_i = p^2r_0 + pu_i + v_j$, $u_iv_j = v_ju_i = p^2r_0 + pu_i + pv_j + w_k$, $u_iw_k = w_ku_i = p^2r_0 + pu_i + pw_k + y_l$, $u_iy_l = y_lu_i = p^2r_0 + pu_i$, $v_jv_{j'} = v_{j'}v_j = p^2r_0 + pv_j + y_l$, $v_jw_k = w_kv_j = p^2r_0 + pv_j + pw_k$, $v_jy_l = y_lv_j = p^2r_0 + pv_j$, $w_kw_{k'} = w_{k'}w_k = p^2r_0 + pw_k$, $w_ky_l = y_lw_k = p^2r_0 + pw_k$, $y_ly_{l'} = y_{l'}y_l = p^2r_0$

Further $u_iu_{i'}u_{i''}u_{i'''}u_{i^{iv}} = 0$, $u_ir_0 = r_0u_i$, $v_jr_0 = r_0v_j$, $w_kr_0 = r_0w_k$, $y_lr_0 = r_0y_l$, where $r_0 \in R_0$ and $1 \le i, i' \le e$, $1 \le j, j' \le f$, $1 \le k, k' \le g$, $1 \le l, l' \le h$. From the given multiplication in R, we see that if $r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l$ and $r_0' + \sum_{i=1}^{e} r_i'u_i + \sum_{j=1}^{f} s_j'v_j + \sum_{k=1}^{g} t_k'w_k + \sum_{l=1}^{h} z_l'y_l$ are any two elements of R, then

$$\begin{aligned}
&\left(r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l\right)\\
&\cdot\left(r_0' + \sum_{i=1}^{e} r_i'u_i + \sum_{j=1}^{f} s_j'v_j + \sum_{k=1}^{g} t_k'w_k + \sum_{l=1}^{h} z_l'y_l\right)\\
&= r_0r_0' + p^a\sum_{i,m=1}^{e}\left(r_ir_m' + pR_0\right) + \sum_{i=1}^{e}\left[r_0r_i' + r_ir_0' + pR_0\right]u_i\\
&+ \sum_{j=1}^{f}\left[(r_0 + pR_0)s_j' + s_j\left(r_0' + pR_0\right) + \sum_{\nu,\mu=1}^{e}\left(r_\nu r_\mu' + pR_0\right)\right]v_j\\
&+ \sum_{k=1}^{g}\left[(r_0 + pR_0)t_k' + t_k\left(r_0' + pR_0\right) + \sum_{i,j}(r_i + pR_0)s_j'\right.\\
&\left.+ s_j\left(r_i' + pR_0\right)\right]w_k + \sum_{l=1}^{h}\left[(r_0 + pR_0)z_l' + z_l\left(r_0' + pR_0\right)\right.\\
&\left.+ \sum_{i,k}(r_i + pR_0)t_k' + t_k\left(r_i' + pR_0\right) + \sum_{\kappa,\tau=1}^{f}\left(s_\kappa s_\tau' + pR_0\right)\right]y_l,
\end{aligned} \tag{1}$$

where $a = 1, 2, 3,$ or 4 depending on whether Char $R_0 = p^2$, p^3, p^4, or p^5. It can be verified that this multiplication turns R into a commutative ring with identity 1.

Notice that if $R_0 = GR(p^r, p)$ where Char $R = p$, then the above multiplication reduces to

$$\begin{aligned}
&\left(r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l\right)\\
&\cdot\left(r_0' + \sum_{i=1}^{e} r_i'u_i + \sum_{j=1}^{f} s_j'v_j + \sum_{k=1}^{g} t_k'w_k + \sum_{l=1}^{h} z_l'y_l\right)\\
&= r_0r_0' + \sum_{i=1}^{e}\left[r_0r_i' + r_ir_0'\right]u_i + \sum_{j=1}^{f}\left[(r_0)s_j' + s_j\left(r_0'\right) + \sum_{\nu,\mu=1}^{e}\left(r_\nu r_\mu'\right)\right]v_j\\
&+ \sum_{k=1}^{g}\left[(r_0)t_k' + t_k\left(r_0'\right) + \sum_{i,j}(r_i)s_j' + s_j\left(r_i'\right)\right]w_k\\
&+ \sum_{l=1}^{h}\left[(r_0)z_l' + z_l\left(r_0'\right) + \sum_{i,k}(r_i)t_k' + t_k\left(r_i'\right) + \sum_{\kappa,\tau=1}^{f}\left(s_\kappa s_\tau'\right)\right]y_l.
\end{aligned} \tag{2}$$

Since the ring

$$\begin{aligned}
R = R_0 \oplus \sum_{i=1}^{e} R_0u_i \oplus \sum_{j=1}^{f} R_0v_j \oplus \sum_{k=1}^{g} R_0w_k\\
\oplus \sum_{l=1}^{h} R_0y_l (\text{as } R_0\text{-module}),
\end{aligned} \tag{3}$$

we have its maximal ideal as

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$$Z(R) = pR_0 \oplus \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l, \quad (4)$$

and the finite abelian p-group given by

$$1 + Z(R) = 1 + pR_0 \oplus \sum_{i=1}^{e} R_0 u_i \oplus \sum_{j=1}^{f} R_0 v_j \oplus \sum_{k=1}^{g} R_0 w_k \oplus \sum_{l=1}^{h} R_0 y_l. \quad (5)$$

The unit groups of R is given by R^*, and it can be verified that

$$R^* = (R^*/1 + Z(R)) \times (1 + Z(R)) = <b> \times (1 + Z(R)), \quad (6)$$

where $<b>$ is a cyclic group of order $p^r - 1$. The structure problem of R^* is reduced to that of $1 + Z(R)$. Using the ideas of Raghavendran [11] and Chikunji [1], we classify the unit groups of the rings constructed in this section.

Proposition 1. *Let R be a ring constructed above and of characteristic p with $pu_i = 0$, $pv_j = 0$, $pw_k = 0$, and $py_l = 0$. Then, its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_2^r)^g, & \text{if } p = 2, \\ \mathbb{Z}_{3^r-1} \times (\mathbb{Z}_9^r)^e \times (\mathbb{Z}_3^r)^f \times (\mathbb{Z}_3^r)^h, & \text{if } p = 3, \\ \mathbb{Z}_{p^r-1} \times \left(\mathbb{Z}_p^r\right)^e \times \left(\mathbb{Z}_p^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & \text{if } p > 3. \end{cases} \quad (7)$$

Proof. See Proposition 3.1 in [5]. □

3. Main Results

We now investigate the structures of the unit groups of finite completely primary rings with a maximal ideals $Z(R)$ such that $(Z(R))^5 = (0)$, $(Z(R))^4 \neq (0)$ and with the characteristics $p^k, 1 \leq k \leq 5$, under the restrictions $p^{\psi} u_i = 0$, $p^{\zeta} v_j = 0$, $pw_k = 0$, $py_l = 0$, where $\psi = 2, 3, 4$; $\zeta = 2, 3$; $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, and $1 \leq l \leq h$.

Proposition 2. *Let R be a ring constructed above and of characteristic p^2 with $p^2 u_i = 0$, $p^2 v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times \left(\mathbb{Z}_{p^2}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \quad (8)$$

Proof. Since R is commutative, $R^* = <b> \cdot (1 + Z(R)) \cong <b> \times (1 + Z(R))$, a direct product of the p-group $1 + Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1 + Z(R)$ of the unit group R^*. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. We consider the two cases separately. □

Case 1. p is even. For each $t = 1, \cdots, r$, $(1 + 2\xi_t)^2 = 1$, $(1 + \xi_t u_i)^8 = 1$, $(1 + \xi_t v_j)^4 = 1$, $(1 + \xi_t w_k)^2 = 1$. For nonnegative integers α_t, δ_t, φ_t, and λ_t with $\alpha_t \leq 2$, $\delta_t \leq 8$, $\varphi_t \leq 4$, and $\lambda_t \leq 2$, it is clear that

$$\prod_{t=1}^{r} \left\{(1 + 2\xi_t)^{\alpha_t}\right\} \cdot \prod_{i=1}^{e} \prod_{t=1}^{r} \left\{(1 + \xi_t u_i)^{\delta_t}\right\} \cdot \prod_{j=1}^{f} \prod_{t=1}^{r} \left\{(1 + \xi_t v_j)^{\varphi_t}\right\} \cdot \prod_{k=1}^{g} \prod_{t=1}^{r} \left\{(1 + \xi_t w_k)^{\lambda_t}\right\} = \{1\} \quad (9)$$

will imply that $\alpha_t = 2$, $\delta_t = 8$, $\varphi_t = 4$, and $\lambda_t = 2$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1 + 2\xi_t)^{\alpha} : \alpha = 1, 2; \forall t = 1, \cdots, r\}$; $B_{t,i} = \{(1 + \xi_t u_i)^{\delta} : \delta = 1, \cdots, 8; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $C_{t,j} = \{(1 + \xi_t v_j)^{\varphi} : \varphi = 1, \cdots, 4; \forall t = 1, \cdots, r\}$, $(j = 1, \cdots, f)$; and $D_{t,k} = \{(1 + \xi_t w_k)^{\lambda} : \lambda = 1, 2; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; we see that A_t, $B_{t,i}$, $C_{t,j}$, and $D_{t,k}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1 + 2\xi_t>$, $<1 + \xi_t u_i>$, $<1 + \xi_t v_j>$, and $<1 + \xi_t w_k>$ is trivial and that

$$\left|\prod_{t=1}^{r} <1 + 2\xi_t>\right| \cdot \left|\prod_{i=1}^{e} \prod_{t=1}^{r} <1 + \xi_t u_i>\right| \cdot \left|\prod_{j=1}^{f} \prod_{t=1}^{r} <1 + \xi_t v_j>\right| \cdot \left|\prod_{k=1}^{g} \prod_{t=1}^{r} <1 + \xi_t w_k>\right| \quad (10)$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} <1 + 2\xi_t> \times \prod_{i=1}^{e} \prod_{t=1}^{r} <1 + \xi_t u_i> \\ &\quad \times \prod_{j=1}^{f} \prod_{t=1}^{r} <1 + \xi_t v_j> \times \prod_{k=1}^{g} \prod_{t=1}^{r} <1 + \xi_t w_k> \\ &\cong \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g. \end{aligned} \quad (11)$$

Case 2. p is odd. For each $t = 1, \cdots, r$, $(1 + p\xi_t)^p = 1$, $(1 + \xi_t u_i)^{p^2} = 1$, $(1 + \xi_t v_j)^{p^2} = 1$, $(1 + \xi_t w_k)^p = 1$, and $(1 + \xi_t y_l)^p = 1$. For nonnegative integers β_t, α_t,δ_t, φ_t, and λ_t with $\beta_t \leq p$, $\alpha_t \leq p^2$, $\delta_t \leq p^2$, $\varphi_t \leq p$, and $\lambda_t \leq p$, it is clear that

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$$\prod_{t=1}^{r}\left\{(1+p\xi_t)^{\beta_t}\right\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\xi_t u_i)^{\alpha_t}\right\}\cdot\prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\xi_t v_j)^{\delta_t}\right\}\cdot\prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\xi_t w_k)^{\varphi_t}\right\}\cdot\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\xi_t y_l)^{\lambda_t}\right\}=\{1\} \tag{12}$$

will imply that $\beta_t = p$, $\alpha_t = p^2$, $\delta_t = p^2$, $\varphi_t = p$, and $\lambda_t = p$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1+p\xi_t)^{\beta} : \beta = 1, \cdots, p; \forall t = 1, \cdots, r\}$; $B_{t,i} = \{(1+\xi_t u_i)^{\alpha} : \alpha = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$ $(i = 1, \cdots, e)$; $C_{t,j} = \{(1+\xi_t v_j)^{\delta} : \delta = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$, $(j = 1, \cdots, f)$; $D_{t,k} = \{(1+\xi_t w_k)^{\varphi} : \varphi = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; and $E_{t,l} = \{(1+\xi_t y_l)^{\lambda} : \lambda = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h)$, we see that A_t, $B_{t,i}$, $C_{t,j}$, $D_{t,k}$, and $E_{t,l}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+p\xi_t>$, $<1+\xi_t u_i>$, $<1+\xi_t v_j>$, $<1+\xi_t w_k>$, and $<1+\xi_t y_l>$ is trivial and that

$$\left|\prod_{t=1}^{r}<1+p\xi_t>\right|\cdot\left|\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\right|\cdot\left|\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\right|\cdot\left|\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\right|\cdot\left|\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\right| \tag{13}$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r}<1+p\xi_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\\ &\quad\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\\ &\cong \mathbb{Z}_p^r\times\left(\mathbb{Z}_{p^2}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h.\end{aligned} \tag{14}$$

Proposition 3. *Let R be a ring constructed above and of characteristic p^3 with $p^2u_i = 0$, $p^2v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^r-1}\times\mathbb{Z}_2^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_2^r)^e\times(\mathbb{Z}_8^r)^{e+f}\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h, & p=2,\\ \mathbb{Z}_{p^r-1}\times\mathbb{Z}_{p^2}^r\times\left(\mathbb{Z}_{p^2}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, & p\neq 2.\end{cases} \tag{15}$$

Proof. Since R is commutative, $R^* = <b>\cdot(1+Z(R)) \cong <b> \times (1+Z(R))$, a direct product of the p-group $1+Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. We consider the two cases separately. □

Case 1. p is even. For each $t = 1, \cdots, r$, $(1+2\xi_t)^2 = 1$, $(1+4\xi_t)^2 = 1$, $(1+2\xi_t u_i)^2 = 1$, $(1+\xi_t u_i + \xi_t v_j)^8 = 1$, $(1+\xi_t w_k)^2 = 1$, and $(1+\xi_t y_l)^2 = 1$. For nonnegative integers η_t, β_t, α_t, δ_t, φ_t, and λ_t with $\eta_t \le 2$, $\beta_t \le 2$, $\alpha_t \le 2$, $\delta_t \le 8$, $\varphi_t \le 2$, and $\lambda_t \le 2$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+2\xi_t)^{\eta_t}\right\}\cdot\prod_{t=1}^{r}\left\{(1+4\xi_t)^{\beta_t}\right\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+2\xi_t u_i)^{\alpha_t}\right\}\cdot\prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\xi_t u_i+\xi_t v_j)^{\delta_t}\right\}\cdot\prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\xi_t w_k)^{\varphi_t}\right\}\cdot\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\xi_t y_l)^{\lambda_t}\right\}=\{1\} \tag{16}$$

will imply that $\eta_t = 2$, $\beta_t = 2$, $\alpha_t = 2$, $\delta_t = 8$, $\varphi_t = 2$, and $\lambda_t = 2$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1+2\xi_t)^{\eta} : \eta = 1, 2; \forall t = 1, \cdots, r\}$; $B_t = \{(1+4\xi_t)^{\beta} : \beta = 1, 2; \forall t = 1, \cdots, r\}$; $C_{t,i} = \{(1+2\xi_t u_i)^{\alpha} : \alpha = 1, 2; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $D_{t,i,j} = \{(1+\xi_t u_i + \xi_t v_j)^{\delta} : \delta = 1, \cdots, 8; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$, $(j = 1, \cdots, f)$; $E_{t,k} = \{(1+\xi_t w_k)^{\varphi} : \varphi = 1, 2; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; $F_{t,l} = \{(1+\xi_t y_l)^{\lambda} : \lambda = 1, 2; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h)$; we see that A_t, B_t, $C_{t,i}$, $D_{t,i,j}$, $E_{t,k}$, and $F_{t,l}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $\langle 1+2\xi_t\rangle$, $\langle 1+4\xi_t\rangle$, $\langle 1+2\xi_t u_i\rangle$, $\langle 1+\xi_t u_i+\xi_t v_j\rangle$, $\langle 1+\xi_t w_k\rangle$, and $\langle 1+\xi_t y_l\rangle$ is trivial and that

$$\left|\prod_{t=1}^{r}<1+2\xi_t>\right|\cdot\left|\prod_{t=1}^{r}<1+4\xi_t>\right|\cdot\left|\prod_{i=1}^{e}\prod_{t=1}^{r}<1+2\xi_t u_i>\right|\cdot\left|\prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i+\xi_t v_j>\right|\cdot\left|\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\right|\cdot\left|\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\right| \tag{17}$$

coincides with $|1 + Z(R)|$, it follows that

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$$1+Z(R)=\prod_{t=1}^{r}<1+2\xi_t>\times\prod_{t=1}^{r}<1+4\xi_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+2\xi_t u_i>\times\prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i+\xi_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\cong\mathbb{Z}_2^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_2^r)^e\times(\mathbb{Z}_8^r)^{e+f}\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h. \tag{18}$$

Case 2. p is odd. For each $t=1,\cdots,r$, $(1+p\xi_t)^{p^2}=1$, $(1+\xi_t u_i)^{p^2}=1$, $(1+\xi_t v_j)^{p^2}=1$, $(1+\xi_t w_k)^p=1$, and $(1+\xi_t y_l)^p=1$. For nonnegative integers β_t, α_t, δ_t, φ_t, and λ_t with $\beta_t\le p^2$, $\alpha_t\le p^2$, $\delta_t\le p^2$, $\varphi_t\le p$, and $\lambda_t\le p$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+p\xi_t)^{\beta_t}\right\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\xi_t u_i)^{\alpha_t}\right\}\cdot\prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\xi_t v_j)^{\delta_t}\right\}\cdot\prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\xi_t w_k)^{\varphi_t}\right\}\cdot\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\xi_t y_l)^{\lambda_t}\right\}=\{1\} \tag{19}$$

will imply that $\beta_t=p^2$, $\alpha_t=p^2$, $\delta_t=p^2$, $\varphi_t=p$, and $\lambda_t=p$, for all $t=1,\cdots,r$.

If we set $A_t=\{(1+p\xi_t)^{\beta}:\beta=1,\cdots,p^2;\forall t=1,\cdots,r\}$; $B_{t,i}=\{(1+\xi_t u_i)^{\alpha}:\alpha=1,\cdots,p^2;\forall t=1,\cdots,r\}$, $(i=1,\cdots,e)$; $C_{t,j}=\{(1+\xi_t v_j)^{\delta}:\delta=1,\cdots,p^2;\forall t=1,\cdots,r\}$, $(j=1,\cdots,f)$; $D_{t,k}=\{(1+\xi_t w_k)^{\varphi}:\varphi=1,\cdots,p;\forall t=1,\cdots,r\}$, $(k=1,\cdots,g)$; and $E_{t,l}=\{(1+\xi_t y_l)^{\lambda}:\lambda=1,\cdots,p;\forall t=1,\cdots,r\}$, $(l=1,\cdots,h)$, we see that A_t, $B_{t,i}$, $C_{t,j}D_{t,k}$, and $E_{t,l}$ are all cyclic subgroups of $1+Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $\langle 1+p\xi_t\rangle$, $\langle 1+\xi_t u_i\rangle$, $\langle 1+\xi_t v_j\rangle$, $\langle 1+\xi_t w_k\rangle$, and $\langle 1+\xi_t y_l\rangle$ is trivial and that

$$\left|\prod_{t=1}^{r}<1+p\xi_t>\right|\cdot\left|\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\right|\cdot\left|\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\right|\cdot\left|\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\right|\cdot\left|\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\right| \tag{20}$$

coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+p\xi_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\cong\mathbb{Z}_{p^2}^r\times\left(\mathbb{Z}_{p^2}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h. \tag{21}$$

Proposition 4. *Let R be a ring constructed above and of characteristic p^3 with $p^3u_i=0$, $p^2v_j=0$, $pw_k=0$, and $py_l=0$. Then its group of units is characterized as follows:*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^{r-1}}\times\mathbb{Z}_2^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h, & p=2,\\ \mathbb{Z}_{p^{r-1}}\times\mathbb{Z}_{p^2}^r\times\left(\mathbb{Z}_{p^3}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, & p\neq 2.\end{cases} \tag{22}$$

Proof. Since $R^*\cong\mathbb{Z}_{p^{r-1}}\times(1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\xi_1,\cdots,\xi_r$ be elements of R_0 with $\xi_1=1$ so that $\bar{\xi}_1,\cdots,\bar{\xi}_r\in R_0/pR_0\cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1\le t\le r$, $1\le i\le e$, $1\le j\le f$, $1\le k\le g$, $1\le l\le h$, the generators are $1+2\xi_t$ of order 2; $1+4\xi_t$ of order 2; $1+\xi_t u_i$ of order 8; $1+\xi_t v_j$ of order 4; $1+\xi_t w_k$ of order 2; and $1+\xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Case 2. p is odd; $1\le t\le r$, $1\le i\le e$, $1\le j\le f$, $1\le k\le g$, $1\le l\le h$, the generators are $1+p\xi_t$ of order p^2; $1+\xi_t u_i$ of order p^3; $1+\xi_t v_j$ of order p^2; $1+\xi_t w_k$ of order p; and $1+\xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Proposition 5. *Let R be a ring constructed above and of characteristic p^3 with $p^3u_i=0$, $p^3v_j=0$, $pw_k=0$, and $py_l=0$. Then its group of units is characterized as follows:*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^{r-1}}\times\mathbb{Z}_2^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_4^r)^e\times(\mathbb{Z}_{16}^r)^{e+f}\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h, & p=2,\\ \mathbb{Z}_{p^{r-1}}\times\mathbb{Z}_{p^2}^r\times\left(\mathbb{Z}_{p^3}^r\right)^e\times\left(\mathbb{Z}_{p^3}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, & p\neq 2.\end{cases} \tag{23}$$

Proof. Since $R^*\cong\mathbb{Z}_{p^{r-1}}\times(1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\xi_1,\cdots,\xi_r$ be elements of R_0 with $\xi_1=1$ so that $\bar{\xi}_1,\cdots,\bar{\xi}_r\in R_0/pR_0\cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

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Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 2; $1 + 4\xi_t$ of order 2; $1 + 2\xi_t u_i$ of order 4; $1 + \xi_t u_i + \xi_t v_j$ of order 16 ; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^2; $1 + \xi_t u_i$ of order p^3; $1 + \xi_t v_j$ of order p^3; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Proposition 6. *Let R be a ring constructed above and of characteristic p^4 with $p^2 u_i = 0$, $p^2 v_j = 0$, $p w_k = 0$, and $p y_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times \left(\mathbb{Z}_{p^2}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \ne 2. \end{cases} \tag{24}$$

Proof. Since R is commutative, $R^* = <b> \cdot (1 + Z(R)) \cong <b> \times (1 + Z(R))$, a direct product of the p-group $1 + Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1 + Z(R)$ of the unit group R^*. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. We consider the two cases separately. □

Case 1. p is even. For each $t = 1, \cdots, r$, $(1 + 2\xi_t)^4 = 1$, $(1 + 6\xi_t)^2 = 1$, $(1 + 2\xi_t u_i)^2 = 1$, $(1 + \xi_t u_i + \xi_t v_j)^8 = 1$, $(1 + \xi_t w_k)^2 = 1$, and $(1 + \xi_t y_l)^2 = 1$. For nonnegative integers η_t, β_t, α_t, δ_t, φ_t, and λ_t with $\eta_t \le 4$, $\beta_t \le 2$, $\alpha_t \le 2$, $\delta_t \le 8$, $\varphi_t \le 2$, and $\lambda_t \le 2$, it is clear that

$$\begin{aligned} &\prod_{t=1}^{r} \left\{(1 + 2\xi_t)^{\eta_t}\right\} \cdot \prod_{t=1}^{r} \left\{(1 + 6\xi_t)^{\beta_t}\right\} \cdot \prod_{i=1}^{e} \prod_{t=1}^{r} \left\{(1 + 2\xi_t u_i)^{\alpha_t}\right\} \\ &\cdot \prod_{j=1}^{f} \prod_{i=1}^{e} \prod_{t=1}^{r} \left\{(1 + \xi_t u_i + \xi_t v_j)^{\delta_t}\right\} \\ &\cdot \prod_{k=1}^{g} \prod_{t=1}^{r} \left\{(1 + \xi_t w_k)^{\varphi_t}\right\} \cdot \prod_{l=1}^{h} \prod_{t=1}^{r} \left\{(1 + \xi_t y_l)^{\lambda_t}\right\} = \{1\} \end{aligned} \tag{25}$$

will imply that $\eta_t = 4$, $\beta_t = 2$, $\alpha_t = 2$, $\delta_t = 8$, $\varphi_t = 2$, and $\lambda_t = 2$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1 + 2\xi_t)^\eta : \eta = 1, \cdots, 4; \forall t = 1, \cdots, r\}$; $B_t = \{(1 + 6\xi_t)^\beta : \beta = 1, 2; \forall t = 1, \cdots, r\}$; $C_{t,i} = \{(1 + 2\xi_t u_i)^\alpha : \alpha = 1, 2; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $D_{t,i,j} = \{(1 + \xi_t u_i + \xi_t v_j)^\delta : \delta = 1, \cdots, 8; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$, $(j = 1, \cdots, f)$; $E_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, 2; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; and $F_{t,l} = \{(1 + \xi_t y_l)^\lambda : \lambda = 1, 2; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h)$, we see that A_t, B_t, $C_{t,i}$, $D_{t,i,j}$, $E_{t,k}$, and $F_{t,l}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1 + 2\xi_t>$, $<1 + 6\xi_t>$, $<1 + 2\xi_t u_i>$, $<1 + \xi_t u_i + \xi_t v_j>$, $<1 + \xi_t w_k>$, and $<1 + \xi_t y_l>$ is trivial and that

$$\begin{aligned} &\left|\prod_{t=1}^{r} <1 + 2\xi_t>\right| \cdot \left|\prod_{t=1}^{r} <1 + 6\xi_t>\right| \\ &\cdot \left|\prod_{i=1}^{e} \prod_{t=1}^{r} <1 + 2\xi_t u_i>\right| \cdot \left|\prod_{j=1}^{f} \prod_{i=1}^{e} \prod_{t=1}^{r} <1 + \xi_t u_i + \xi_t v_j>\right| \\ &\cdot \left|\prod_{k=1}^{g} \prod_{t=1}^{r} <1 + \xi_t w_k>\right| \cdot \left|\prod_{l=1}^{h} \prod_{t=1}^{r} <1 + \xi_t y_l>\right| \end{aligned} \tag{26}$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) = &\prod_{t=1}^{r} <1 + 2\xi_t> \times \prod_{t=1}^{r} <1 + 6\xi_t> \\ &\times \prod_{i=1}^{e} \prod_{t=1}^{r} <1 + 2\xi_t u_i> \times \prod_{j=1}^{f} \prod_{i=1}^{e} \prod_{t=1}^{r} <1 \\ &+ \xi_t u_i + \xi_t v_j> \times \prod_{k=1}^{g} \prod_{t=1}^{r} <1 + \xi_t w_k> \\ &\times \prod_{l=1}^{h} \prod_{t=1}^{r} <1 + \xi_t y_l> \cong \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \\ &\times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h. \end{aligned} \tag{27}$$

Case 2. p is odd. For each $t = 1, \cdots, r$, $(1 + p\xi_t)^{p^3} = 1$, $(1 + \xi_t u_i)^{p^2} = 1$, $(1 + \xi_t v_j)^{p^2} = 1$, $(1 + \xi_t w_k)^p = 1$, and $(1 + \xi_t y_l)^p = 1$. For nonnegative integers β_t, α_t, δ_t, φ_t, and λ_t with $\beta_t \le p^3$, $\alpha_t \le p^2$, $\delta_t \le p^2$, $\varphi_t \le p$, and $\lambda_t \le p$, it is clear that

$$\begin{aligned} &\prod_{t=1}^{r} \left\{(1 + p\xi_t)^{\beta_t}\right\} \cdot \prod_{i=1}^{e} \prod_{t=1}^{r} \left\{(1 + \xi_t u_i)^{\alpha_t}\right\} \\ &\cdot \prod_{j=1}^{f} \prod_{t=1}^{r} \left\{(1 + \xi_t v_j)^{\delta_t}\right\} \cdot \prod_{k=1}^{g} \prod_{t=1}^{r} \left\{(1 + \xi_t w_k)^{\varphi_t}\right\} \\ &\cdot \prod_{l=1}^{h} \prod_{t=1}^{r} \left\{(1 + \xi_t y_l)^{\lambda_t}\right\} = \{1\} \end{aligned} \tag{28}$$

will imply that $\beta_t = p^3$, $\alpha_t = p^2$, $\delta_t = p^2$, $\varphi_t = p$, and $\lambda_t = p$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1 + p\xi_t)^\beta : \beta = 1, \cdots, p^3; \forall t = 1, \cdots, r\}$; $B_{t,i} = \{(1 + \xi_t u_i)^\alpha : \alpha = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $C_{t,j} = \{(1 + \xi_t v_j)^\delta : \delta = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$, $(j = 1, \cdots, f)$; $D_{t,k} = \{(1 + \xi_t w_k)^\varphi : \varphi = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$;

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and $E_{t,l} = \{(1+\xi_t y_l)^{\lambda} : \lambda = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h$), we see that A_t, $B_{t,i}$, $C_{t,j}$, $D_{t,k}$, and $E_{t,l}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $\langle 1+p\xi_t\rangle$, $\langle 1+\xi_t u_i\rangle$, $\langle 1+\xi_t v_j\rangle$, $\langle 1+\xi_t w_k\rangle$, and $\langle 1+\xi_t y_l\rangle$ is trivial and that

$$\left|\prod_{t=1}^{r} < 1+p\xi_t > \right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\xi_t u_i > \right| \cdot \left|\prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\xi_t v_j > \right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\xi_t w_k > \right| \cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\xi_t y_l > \right| \tag{29}$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1+Z(R) = {} & \prod_{t=1}^{r} < 1+p\xi_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1+\xi_t u_i > \\ & \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1+\xi_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1+\xi_t w_k > \\ & \times \prod_{l=1}^{h}\prod_{t=1}^{r} < 1+\xi_t y_l > \cong \mathbb{Z}_{p^3}^{r} \times \left(\mathbb{Z}_{p^2}^{r}\right)^{e} \times \left(\mathbb{Z}_{p^2}^{r}\right)^{f} \\ & \times \left(\mathbb{Z}_{p}^{r}\right)^{g} \times \left(\mathbb{Z}_{p}^{r}\right)^{h}. \end{aligned} \tag{30}$$

Proposition 7. *Let R be a ring constructed above and of characteristic p^4 with $p^3u_i = 0$, $p^2v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times \left(\mathbb{Z}_{p^3}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{31}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 4; $1 + 14\xi_t$ of order 2; $1 + \xi_t u_i$ of order 8; $1 + \xi_t v_j$ of order 4; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^3; $1 + \xi_t u_i$ of order p^3; $1 + \xi_t v_j$ of order p^2; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Proposition 8. *Let R be a ring constructed above and of characteristic p^4 with $p^3u_i = 0$, $p^3v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_4^r)^e \times (\mathbb{Z}_{16}^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times \left(\mathbb{Z}_{p^3}^r\right)^e \times \left(\mathbb{Z}_{p^3}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{32}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 4; $1 + 6\xi_t$ of order 2; $1 + 2\xi_t u_i$ of order 4; $1 + \xi_t u_i + \xi_t v_j$ of order 16 ; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^3; $1 + \xi_t u_i$ of order p^3; $1 + \xi_t v_j$ of order p^3; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Proposition 9. *Let R be a ring constructed above and of characteristic p^4 with $p^4u_i = 0$, $p^2v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times \left(\mathbb{Z}_{p^4}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{33}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 4; $1 + 6\xi_t$ of order 2; $1 + \xi_t u_i$ of order 16; $1 + \xi_t v_j$ of order 4; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^3; $1 + \xi_t u_i$ of order

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p^4; $1+\xi_t v_j$ of order p^2; $1+\xi_t w_k$ of order p; and $1+\xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Proposition 10. *Let R be a ring constructed above and of characteristic p^4 with $p^4 u_i = 0$, $p^3 v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_8^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times \left(\mathbb{Z}_{p^4}^r\right)^e \times \left(\mathbb{Z}_{p^3}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{34}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1+Z(R))$, it suffices to determine the structure of $1+Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1+2\xi_t$ of order 4; $1+6\xi_t$ of order 2; $1+\xi_t u_i$ of order 16; $1+\xi_t v_j$ of order 8; $1+\xi_t w_k$ of order 2; and $1+\xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1+p\xi_t$ of order p^3; $1+\xi_t u_i$ of order p^4; $1+\xi_t v_j$ of order p^3; $1+\xi_t w_k$ of order p; and $1+\xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 6.

Proposition 11. *Let R be a ring constructed above and of characteristic p^5 with $p^2 u_i = 0$, $p^2 v_j = 0$, $pw_k = 0$, and$py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^2}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{35}$$

Proof. Since R is commutative, $R^* = <b> \cdot (1+Z(R)) \cong <b> \times (1+Z(R))$, a direct product of the p-group $1+Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. We consider the two cases separately. □

Case 1. p is even. For each $t = 1, \cdots, r$, $(1+2\xi_t)^8 = 1$, $(1+14\xi_t)^2 = 1$, $(1+2\xi_t u_i)^2 = 1$, $(1+\xi_t u_i + \xi_t v_j)^8 = 1$, $(1+\xi_t w_k)^2 = 1$, and $(1+\xi_t y_l)^2 = 1$. For nonnegative integers η_t, β_t, α_t, δ_t, φ_t, and λ_t with $\eta_t \le 8$, $\beta_t \le 2$, $\alpha_t \le 2$, $\delta_t \le 8$, $\varphi_t \le 2$, and $\lambda_t \le 2$, it is clear that

$$\begin{aligned} &\prod_{t=1}^{r} \left\{(1+2\xi_t)^{\eta_t}\right\} \cdot \prod_{t=1}^{r} \left\{(1+14\xi_t)^{\beta_t}\right\} \cdot \prod_{i=1}^{e}\prod_{t=1}^{r} \left\{(1+2\xi_t u_i)^{\alpha_t}\right\} \\ &\cdot \prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r} \left\{(1+\xi_t u_i + \xi_t v_j)^{\delta_t}\right\} \\ &\cdot \prod_{k=1}^{g}\prod_{t=1}^{r} \left\{(1+\xi_t w_k)^{\varphi_t}\right\} \cdot \prod_{l=1}^{h}\prod_{t=1}^{r} \left\{(1+\xi_t y_l)^{\lambda_t}\right\} = \{1\} \end{aligned} \tag{36}$$

will imply that $\eta_t = 8$, $\beta_t = 2$, $\alpha_t = 2$, $\delta_t = 8$, $\varphi_t = 2$, and $\lambda_t = 2$ for all $t = 1, \cdots, r$.

If we set $A_t = \{(1+2\xi_t)^\eta : \eta = 1, \cdots, 8; \forall t = 1, \cdots, r\}$; $B_t = \{(1+14\xi_t)^\beta : \beta = 1, 2; \forall t = 1, \cdots, r\}$; $C_{t,i} = \{(1+2\xi_t u_i)^\alpha : \alpha = 1, 2; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $D_{t,i,j} = \{(1+\xi_t u_i + \xi_t v_j)^\delta : \delta = 1, \cdots, 8; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$, $(j = 1, \cdots, f)$; $E_{t,k} = \{(1+\xi_t w_k)^\varphi : \varphi = 1, 2; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; and $F_{t,l} = \{(1+\xi_t y_l)^\lambda : \lambda = 1, 2; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h)$, we see that A_t, B_t, $C_{t,i}$, $D_{t,i,j}$, $E_{t,k}$, and $F_{t,l}$ are all cyclic subgroups of $1+Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+2\xi_t>$, $<1+14\xi_t>$, $<1+2\xi_t u_i>$, $<1+\xi_t u_i+\xi_t v_j>$, $<1+\xi_t w_k>$, and $<1+\xi_t y_l>$ is trivial and that

$$\begin{aligned} &\left|\prod_{t=1}^{r} <1+2\xi_t>\right| \cdot \left|\prod_{t=1}^{r} <1+14\xi_t>\right| \cdot \left|\prod_{i=1}^{e}\prod_{t=1}^{r} <1+2\xi_t u_i>\right| \\ &\cdot \left|\prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r} <1+\xi_t u_i+\xi_t v_j>\right| \cdot \left|\prod_{k=1}^{g}\prod_{t=1}^{r} <1+\xi_t w_k>\right| \\ &\cdot \left|\prod_{l=1}^{h}\prod_{t=1}^{r} <1+\xi_t y_l>\right| \end{aligned} \tag{37}$$

coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned} 1+Z(R) &= \prod_{t=1}^{r} <1+2\xi_t> \times \prod_{t=1}^{r} <1+14\xi_t> \\ &\times \prod_{i=1}^{e}\prod_{t=1}^{r} <1+2\xi_t u_i> \times \prod_{j=1}^{f}\prod_{i=1}^{e}\prod_{t=1}^{r} <1+\xi_t u_i+\xi_t v_j> \times \prod_{k=1}^{g}\prod_{t=1}^{r} <1+\xi_t w_k> \\ &\times \prod_{l=1}^{h}\prod_{t=1}^{r} <1+\xi_t y_l> \\ &\cong \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_2^r)^e \times (\mathbb{Z}_8^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h. \end{aligned} \tag{38}$$

Case 2. p is odd. For each $t = 1, \cdots, r$, $(1+p\xi_t)^{p^4} = 1$, $(1+\xi_t u_i)^{p^2} = 1$, $(1+\xi_t v_j)^{p^2} = 1$, $(1+\xi_t w_k)^p = 1$, and

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$(1+\xi_t y_l)^p = 1$. For nonnegative integers β_t, α_t, δ_t, φ_t, and λ_t with $\beta_t \le p^4$, $\alpha_t \le p^2$, $\delta_t \le p^2$, $\varphi_t \le p$, and $\lambda_t \le p$, it is clear that

$$\prod_{t=1}^{r}\left\{(1+p\xi_t)^{\beta_t}\right\}\cdot\prod_{i=1}^{e}\prod_{t=1}^{r}\left\{(1+\xi_t u_i)^{\alpha_t}\right\}\cdot\prod_{j=1}^{f}\prod_{t=1}^{r}\left\{(1+\xi_t v_j)^{\delta_t}\right\}\cdot\prod_{k=1}^{g}\prod_{t=1}^{r}\left\{(1+\xi_t w_k)^{\varphi_t}\right\}\cdot\prod_{l=1}^{h}\prod_{t=1}^{r}\left\{(1+\xi_t y_l)^{\lambda_t}\right\}=\{1\} \tag{39}$$

will imply that $\beta_t = p^4$, $\alpha_t = p^2$, $\delta_t = p^2$, $\varphi_t = p$, and $\lambda_t = p$, for all $t = 1, \cdots, r$.

If we set $A_t = \{(1+p\xi_t)^{\beta} : \beta = 1, \cdots, p^4; \forall t = 1, \cdots, r\}$; $B_{t,i} = \{(1+\xi_t u_i)^{\alpha} : \alpha = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$, $(i = 1, \cdots, e)$; $C_{t,j} = \{(1+\xi_t v_j)^{\delta} : \delta = 1, \cdots, p^2; \forall t = 1, \cdots, r\}$, $(j = 1, \cdots, f)$; $D_{t,k} = \{(1+\xi_t w_k)^{\varphi} : \varphi = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(k = 1, \cdots, g)$; and $E_{t,l} = \{(1+\xi_t y_l)^{\lambda} : \lambda = 1, \cdots, p; \forall t = 1, \cdots, r\}$, $(l = 1, \cdots, h)$, we see that A_t, $B_{t,i}$, $C_{t,j}$, $D_{t,k}$, and $E_{t,l}$ are all cyclic subgroups of $1 + Z(R)$, and these are of the orders inferred from their definition. Since the intersection of any pair of the cyclic subgroups $<1+p\xi_t>$, $<1+\xi_t u_i>$, $<1+\xi_t v_j>$, $<1+\xi_t w_k>$, and $<1+\xi_t y_l>$ is trivial and that

$$\left|\prod_{t=1}^{r}<1+p\xi_t>\right|\cdot\left|\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\right|\cdot\left|\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\right|\cdot\left|\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\right|\cdot\left|\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\right| \tag{40}$$

coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned}1+Z(R) &= \prod_{t=1}^{r}<1+p\xi_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\xi_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\xi_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\xi_t w_k>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\xi_t y_l>\\ &\cong \mathbb{Z}_{p^4}^r\times\left(\mathbb{Z}_{p^2}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h.\end{aligned} \tag{41}$$

Proposition 12. *Let R be a ring constructed above and of characteristic p^5 with $p^3u_i = 0$, $p^2v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases}\mathbb{Z}_{2^{r-1}}\times\mathbb{Z}_8^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h, & p=2,\\ \mathbb{Z}_{p^{r-1}}\times\mathbb{Z}_{p^4}^r\times\left(\mathbb{Z}_{p^3}^r\right)^e\times\left(\mathbb{Z}_{p^2}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, & p\neq 2.\end{cases} \tag{42}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 8; $1 + 14\xi_t$ of order 2; $1 + \xi_t u_i$ of order 8; $1 + \xi_t v_j$ of order 4; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^4; $1 + \xi_t u_i$ of order p^3; $1 + \xi_t v_j$ of order p^2; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Proposition 13. *Let R be a ring constructed above and of characteristic p^5 with $p^3u_i = 0$, $p^3v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases}\mathbb{Z}_{2^{r-1}}\times\mathbb{Z}_8^r\times\mathbb{Z}_2^r\times(\mathbb{Z}_4^r)^e\times(\mathbb{Z}_{16}^r)^{e+f}\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h, & p=2,\\ \mathbb{Z}_{p^{r-1}}\times\mathbb{Z}_{p^4}^r\times\left(\mathbb{Z}_{p^3}^r\right)^e\times\left(\mathbb{Z}_{p^3}^r\right)^f\times\left(\mathbb{Z}_p^r\right)^g\times\left(\mathbb{Z}_p^r\right)^h, & p\neq 2.\end{cases} \tag{43}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^{r-1}} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + 2\xi_t$ of order 8; $1 + 14\xi_t$ of order 2; $1 + 2\xi_t u_i$ of order 4; $1 + \xi_t u_i + \xi_t v_j$ of order 16; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Case 2. p is odd; $1 \le t \le r$, $1 \le i \le e$, $1 \le j \le f$, $1 \le k \le g$, $1 \le l \le h$, the generators are $1 + p\xi_t$ of order p^4; $1 + \xi_t u_i$ of order p^3; $1 + \xi_t v_j$ of order p^3; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 3.

Proposition 14. *Let R be a ring constructed above and of characteristic p^5 with $p^4u_i = 0$, $p^2v_j = 0$, $pw_k = 0$, and $py_l = 0$. Then its group of units is characterized as follows:*

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$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_{16}^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^4}^r\right)^e \times \left(\mathbb{Z}_{p^2}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{44}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1 + 2\xi_t$ of order 8; $1 + 14\xi_t$ of order 2; $1 + \xi_t u_i$ of order 16; $1 + \xi_t v_j$ of order 4; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Case 2. p is odd; $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1 + p\xi_t$ of order p^4; $1 + \xi_t u_i$ of order p^4; $1 + \xi_t v_j$ of order p^2; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Proposition 15. *Let R be a ring constructed above and of characteristic p^5 with $p^4 u_i = 0$, $p^3 v_j = 0$, $p w_k = 0$, and $p y_l = 0$. Then its group of units is characterized as follows:*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_{16}^r)^{e+f} \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & p = 2, \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times \left(\mathbb{Z}_{p^4}^r\right)^e \times \left(\mathbb{Z}_{p^3}^r\right)^f \times \left(\mathbb{Z}_p^r\right)^g \times \left(\mathbb{Z}_p^r\right)^h, & p \neq 2. \end{cases} \tag{45}$$

Proof. Since $R^* \cong \mathbb{Z}_{p^r-1} \times (1 + Z(R))$, it suffices to determine the structure of $1 + Z(R)$. Let $\xi_1, \cdots, \xi_r$ be elements of R_0 with $\xi_1 = 1$ so that $\bar{\xi}_1, \cdots, \bar{\xi}_r \in R_0/pR_0 \cong GF(p^r)$ form a basis for $GF(p^r)$ regarded as a vector space over its prime subfield $GF(p)$. Then the generators with their respective orders are as indicated below. □

Case 1. p is even; $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1 + 2\xi_t$ of order 8; $1 + 14\xi_t$ of order 2; $1 + 2\xi_t u_i$ of order 8; $1 + \xi_t u_i + \xi_t v_j$ of order 16; $1 + \xi_t w_k$ of order 2; and $1 + \xi_t y_l$ of order 2. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

Case 2. p is odd; $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1 + p\xi_t$ of order p^4; $1 + \xi_t u_i$ of order p^4; $1 + \xi_t v_j$ of order p^3; $1 + \xi_t w_k$ of order p; and $1 + \xi_t y_l$ of order p. The rest of the proof follows a similar argument and may be deduced from Proposition 11.

4. Conclusions

The unit groups of some classes of five radical zero commutative completely primary finite rings whose first and second Galois ring module generators are of order p^k, $k = 2, 3, 4$, have been classified in this work. It is evident that the results are in piece when $p = 2$ and $p \geq 3$. Since the unit groups of classes of five radical zero commutative completely primary finite rings in this work have been classified via fundamental theorem of finitely generated abelian group, the use of j-diagram technique by References [8–10] is therefore recommended for possible further study of the unit groups of such rings.

Data Availability

The corresponding author may be consulted for all the data used within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in determining the group of units of these classes of rings and in writing this article. All have read and approved the final manuscript.

References

[1] C. J. Chikunji, "Unit groups of cube radical zero commutative completely primary finite rings," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, Article ID 383080, 2005.

[2] C. J. Chikunji, "On unit groups of completely primary finite rings," *Mathematical Journal of Okayama University*, vol. 50, pp. 149–160, 2008.

[3] C. J. Chikunji, "Groups of units of commutative completely primary finite rings," *African Journal of Pure and Applied Mathematics*, vol. 1, pp. 40–48, 2014.

[4] M. O. Oduor and M. O. Onyango, "Unit groups of some classes of power four radical zero commutative completely primary finite rings," *International Journal of Algebra*, vol. 8, pp. 357–363, 2014.

[5] H. S. Were, M. O. Oduor, and M. N. Gichuki, "Unit groups of classes of five radical zero commutative completely primary finite rings," *Journal of Advances in Mathematics and Computer Science*, vol. 36, no. 8, pp. 137–154, 2021.

[6] R. S. Wilson, "On the structure of finite rings," *Compositio Mathematica*, vol. 26, pp. 79–93, 1973.

[7] H. S. Were, M. O. Oduor, and M. N. Gichuki, "Classication of units of five radical zero completely primary finite rings with variant orders of second Galois ring module generators," *Asian Research Journal of Mathematics*, vol. 18, no. 6, pp. 70–78, 2022.

[8] C. W. Ayoub, "On diagrams for abelian groups," *Journal of Number Theory*, vol. 2, no. 4, pp. 442–458, 1970.

[9] C. W. Ayoub, "On the group of units of certain rings," *Journal of Number Theory*, vol. 4, no. 4, pp. 383–403, 1972.

1469, 2022, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1155/2022/7867431, Wiley Online Library on [24/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[10] S. Alabiad and Y. Alkhamees, "Recapturing the structure of group of units of any finite commutative chain ring," *Symmetry*, vol. 13, no. 2, pp. 307–309, 2021.

[11] R. Raghavendran, "Finite associative rings," *Compositio Mathematica*, vol. 21, no. 2, pp. 195–469, 1969.

[12] M. O. Oduor, C. J. Chikunji, and N. O. Ongati, "Unit groups of k + 1 index radical zero commutative finite rings," *International Journal of Pure and Applied Mathematics*, vol. 57, no. 1, pp. 57–67, 2009.

[13] M. O. Oduor, M. O. Onyango, and M. Eliud, "Units of commutative completely primary finite rings of characteristic p^n," *International Journal of Algebra*, vol. 7, no. 6, pp. 259–266, 2013.

1469, 2022, 1, Downloaded from https://onlinelibrary.wiley.com/doi/10.1155/2022/7867431, Wiley Online Library on [24/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License