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Automorphisms of the Unit Groups of Power Four Radical Zero Finite Commutative Completely Primary Rings

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Abstract

Let G be a group. The groups $G^{'}$ for which G is an automorphism group have not been fully characterized. Suppose R is a class of power four radical zero finite commutative completely primary ring and J its Jacobson Radical, the structure of R^*, the unit groups of R is precisely known, but the automorphisms of these unit groups is unknown. In this paper, we discover the structure and order of these automorphisms for all the characteristics of the ring R.

Mathematics Subject Classification: Primary 20K30; Secondary 16P10

Keywords: Automorphism Groups, Unit Groups, Power four Radical Zero, Completely Primary Rings

1 Introduction

The definition of terms and standard notations can be obtained from [5, 6]. The automorphism group of R^* denoted as $Aut(R^*)$ is a set whose elements are automorphisms $\sigma : R^* \to R^*$ and where the group operation is composition of automorphisms. Thus, its group structure is obtained as a subgroup of the $Sym(R^*)$, the group of all permutations on R^*. Given an arbitrary finite group G, the computation of its automorphism group $Aut(G)$ is not a very easy task. Pioneer work in this regard was carried out by Felsch and Neubuser [7, 8] who developed an algorithm which made use of their subgroup lattice program to compute the automorphisms. In the early 1970s, Neubuser independently developed a technique to determine the automorphism groups by considering its action on the union of certain conjugacy classes of G. Similar methods were used by Hulpke [1], Cannon and Holt [3] who presented a new algorithm to answer this problem.

A few more efficient approaches to determine the automorphism groups of the groups satisfying certain properties are available. Following the work of Shoda [4], Hulpke in 1997 implemented a practical method for finite Abelian groups. Hillar and Rhea [2] determined the order of the automorphisms of an arbitrary finite Abelian group G. Their approach has got a more involved argument. Therefore in this paper, we use the approaches developed in [2] to determine the structure and order of the automorphisms of R^*. Owing to the structure of $1+J$ a subgroup of R^*, we have developed a set of square matrices R_p of order the rank of $1+J$ in all the cases considered. From R_p, we identified all the endomorphisms of $1 + J$ and specified which endomorphisms are the automorphisms of $1 + J$. Finally, we counted all the automorphisms of $1 + J$. Since $R^* = \mathbb{Z}_{p^r-1} \times (1 + J)$, and $g.c.d(\mid \mathbb{Z}_{p^r-1} \mid, \mid 1 + J \mid) = 1$, it easily follows from [2] that $Aut(R^*) \cong Aut(\mathbb{Z}_{p^r-1}) \times Aut(1 + J)$.

2 Units of Power Four Radical Zero Commutative finite Completely Primary Rings

In [6], we constructed some classes of power four radical zero commutative completely primary finite rings and determined their unit groups. In this section, we give detailed recap of the same constructions, demonstrate the

structures of the unit groups whose automorphisms are determined in the next section.

2.1 Rings of Characteristic p

For any prime integer p and a positive integer r, let $R_0 = GR(p^r, p)$ be a Galois ring of order p^r and characteristic p. Suppose U, V and W are finitely generated R_0-modules such that $dim_{R_o}U = s$, $dim_{R_o}V = t$ and $dim_{R_o}W = \lambda$ and $s + t + \lambda = h$. Let $\{u_1, u_2, ...u_s\}$,$\{v_1, v_2, ...v_t\}$ and $\{w_1, w_2, ...w_\lambda\}$ be the generators of U, V, W respectively so that $R = R_0 \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = 1, t = 1, \lambda = h - 2$, so that $R = R_0 \oplus R_0u \oplus R_0v \oplus \sum_{j=1}^{h-2} R_0w_j$ and $pu = pv = pw_j = 0, 1 \leq j \leq h - 2$. On R, define multiplication as follows;

$(r_0, r_1, r_2, ..., r_h)(s_0, s_1, s_2, ..., s_h) = (r_0s_0, r_0s_1+r_1s_0, r_0s_2+r_2s_0+r_1s_1, r_0s_3+ r_3s_0+r_1s_2+r_2s_1), ..., r_0s_h+r_hs_0+r_1s_2+r_2s_1)$. It is easy to verify that the given multiplication turns R into a commutative ring with identity $(1, 0, .., 0)$. Moreover, $J = R_0u \oplus R_0v \oplus \sum_{j=1}^{\lambda} R_0w_j, J^2 = R_0v \oplus \sum_{j=1}^{\lambda} R_0w_j, J^3 = \sum_{j=1}^{\lambda} R_0w_j$ and $J^4 = (0)$.

Lemma 1. *(See prop. 3 in [6]) Let R be the ring constructed above and J be its Jacobson radical. Then*

$$R^* \cong \{ \ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_p^r)^\lambda, \quad \text{if } p \neq 2.$$

2.2 Rings of characteristic p^2

For any prime integer p and positive integer r, let $R_0 = GR(p^{2r}, p^2)$ be a Galois ring of order p^{2r} and characteristic p^2. Suppose U, V and W are finitely generated R_0-modules such that $dim_{R_o}U = s$, $dim_{R_o}V = t$ and $dim_{R_o}W = \lambda$ and $s+t+\lambda = h$. Let $\{u_1, ..., u_s\}$, $\{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of U, V, W respectively so that $R = R_0 \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h-1, t = 1, \lambda = 0$, so that R can be expressed as $R = R_0 \oplus \sum_{j=1}^{h-1} R_0u_j \oplus R_0v$, where $pu_j \neq 0, p^2u_j = 0, 1 \leq j \leq s$ and $pv = 0$. On R, define multiplication as follows;

$(r_0, r_1, r_2, ..., r_{h-1}, \overline{r_h})(s_0, s_1, s_2, ..., s_{h-1}, \overline{s_h}) = (r_0s_0 + p\sum_{i,j=1}^{h-1} r_is_j, r_0s_1 + r_1s_0, ..., r_0s_{h-1} + r_{h-1}s_o, r_o\overline{s_h} + \overline{r_h}s_o)$ where $\overline{r_h}, \overline{s_h} \in R_o/pR_o$. It is easy to verify that the given multiplication turns R into a commutative ring with identity $(1, 0, ...0, \overline{0})$. The ring R constructed is completely primary of characteristic p^2 and $J = pR_0 \oplus \sum_{j=1}^{s} R_0u_j \oplus R_0v, J^2 = pR_0 \oplus p\sum_{j=1}^{s} R_0u_j \oplus R_0v, J^3 = p\sum_{j=1}^{s} R_0u_j$ and $J^4 = (0)$

Lemma 2. *(See prop. 5 in [6]) Let R be the ring constructed and J be*

its Jacobson radical. Then

$$R^* \cong \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_{p^2}^r)^s \times (\mathbb{Z}_p^r)^2$$

for every prime integer p and positive integer r

2.3 Rings of characteristic p^3

For any prime integer p and positive integer r, let $R_0 = GR(p^{3r}, p^3)$ be a Galois ring of order p^{3r} and characteristic p^3. Suppose U, V and W are finitely generated R_0-modules such that $dim_{R_o}U = s$, $dim_{R_o}V = t$ and $dim_{R_o}W = \lambda$ and $s+t+\lambda = h$. Let $\{u_1, ..., u_s\}$, $\{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of U, V, W respectively so that $R = R_0 \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h-1, t = 1, \lambda = 0$, so that R can be expressed as $R = R_0 \oplus \sum_{j=1}^{h-1} R_0u_j \oplus R_0v$, where , $p^2u_j \neq 0, p^3u_j = 0; 1 \leq j \leq s$ and $pv = 0$. On R, define multiplication as follows;

$(r_0, \overline{r_1}, \overline{r_2}, ..., \overline{r_{h-1}}, \widehat{r_h})(s_0, \overline{s_1}, \overline{s_2}, ..., \overline{s_{h-1}}\widehat{s_h}) = (r_0s_0, r_0\overline{s_1} + \overline{r_1}s_0, ..., r_0\overline{s_{h-1}} + \overline{r_{h-1}}s_0, r_0\widehat{s_h} + \widehat{r_h}s_0 + \sum_{i,j=1}^{h-1} \overline{r_is_j})$ where $\overline{r_i}, \overline{s_j} \in R_o/p^2R_o$ and $\widehat{r_h}, \widehat{s_h} \in R_o/pR_o$. It is readily verified that the given multiplication turns R into a commutative ring with identity $(1, \bar{0}, ..., \bar{0}, \widehat{0})$. The ring constructed is completely primary of characteristic p^3 and $J = pR_0 \oplus \sum_{j=1}^{s} R_0u_j \oplus R_0v, J^2 = p^2R_0 \oplus p\sum_{j=1}^{s} R_0u_j \oplus R_0v, J^3 = pR_0v$ and $J^4 = (0)$

Lemma 3. *(See prop. 7 in [6]) Let R be a ring constructed and J be its Jacobson radical, then its group of units is characterized as follows*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_4^r \times (\mathbb{Z}_2^r)^{s-1}, & \text{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \mathbb{Z}_{p^2}^r \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^s, & \text{if } p \neq 2. \end{cases}$$

2.4 Rings of Characteristics p^4

For a prime integer p and a positive integer r, let $R_0 = GR(p^{4r}, p^4)$ be a Galois ring of order p^{4r} and characteristic p^4. Suppose U, V and W are finitely generated R_0-modules such that $dim_{R_o}U = s$, $dim_{R_o}V = t$ and $dim_{R_o}W = \lambda$ and $s+t+\lambda = h$. Let $\{u_1, ..., u_s\}$, $\{v_1, ..., v_t\}$ and $\{w_1, ..., w_\lambda\}$ be the generators of U, V, W respectively so that $R = R_0 \oplus U \oplus V \oplus W$ is an additive abelian group. Further, assume that $s = h$, $t = 0$, $\lambda = 0$ so that $R = R_o\sum_{j=1}^{s} R_ou_j$ where $pu_j = 0$, $1 \leq j \leq s$. On R, define multiplication as follows;

$(r_0, \overline{r_1}, \overline{r_2}, ..., \overline{r_h})(s_0, \overline{s_1}, \overline{s_2}, ..., \overline{s_h}) = (r_0s_0, r_0\overline{s_1}+\overline{r_1}s_0, ..., r_0\overline{s_h}+\overline{r_h}s_0)$ where,$\overline{r_i}, \overline{s_j} \in R_0/pR_0; 1 \leq i, j \leq h$. This multiplication turns R into a commutative ring with identity $(1, \bar{0}, ..., \bar{0})$. The ring constructed is completely primary, of characteristic p^4 with Jacobson radical such that $J = pR_0 \oplus \sum_{j=1}^{s} R_ou_j, J^2 = p^2R_0, J^3 = p^3R_0$ and $J^4 = (0)$

Lemma 4. *(Similar to prop. 9 in [6]) Let R be the ring described by the construction above. Then*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8^{r-1} \times (\mathbb{Z}_2^r)^s & \text{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^r)^s & \text{if } p \neq 2. \end{cases}$$

Theorem 1. *The structure of the units R^* of the commutative completely primary finite ring R of characteristic p, p^2, p^3, p^4 with maximal ideal J such that $J^4 = (0)$ and $J^3 \neq (0)$, with the invariants p, r, s, t, h and λ where $p \in J$, is a direct product of cyclic groups as follows:*

(i) If the $charR = p$ then,

$$R^* \cong \left\{ \ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_p^r)^\lambda, \quad \text{if } p \neq 2. \right.$$

(ii) If the $charR = p^2$ then,

$$R^* \cong \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^2 \times (\mathbb{Z}_{p^2}^r)^s$$

(iii) If the $charR = p^3$ then,

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \mathbb{Z}_8^r \times \mathbb{Z}_4^r \times (\mathbb{Z}_2^r)^{s-1}, & \text{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^s, & \text{if } p \neq 2. \end{cases}$$

(iv) If the $charR = p^4$ then,

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8^{r-1} \times (\mathbb{Z}_2^r)^s & \text{if } p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_p^r)^s & \text{if } p \neq 2. \end{cases}$$

3 Automorphism groups of the units of Power Four Radical zero finite commutative Completely Primary rings

We characterize the structure and order of the automorphism groups of the unit groups whose structures are described in Theorem 1. We pay particular attention to each case by describing the matrix R_p and the endomorphisms $M = \psi(A)$ such that $A \in R_p$ and $\psi : R_p \to End(1 + J)$ is a surjective ring homomorphism.

The following results are important in the sequel,

Proposition 1. *Let* $1+J \cong \mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}}$ *with* $e_1 \leq e_2$. *Then, the matrix* $\begin{pmatrix} i & r \\ j & s \end{pmatrix}$ *represents:*

(i) An Endomorphism of $1+J$ *if and only if* $i \in \mathbb{Z}_{p^{e_1}}$, $j \equiv 0 mod(p^{e_1-e_2})$, $r \in \mathbb{Z}_{p^{e_2}}$ *and* $s \in \mathbb{Z}_{p^{e_2}}$

(ii) An Automorphism of $1+J$ *if and only if* $i \in (\mathbb{Z}_{p^{e_1}})^*$, $j \equiv 0 mod(p^{e_1-e_2})$, $r \in \mathbb{Z}_{p^{e_2}}$ *and* $s \in (\mathbb{Z}_{p^{e_2}})^*$

Lemma 5. *Let* R *be a class of ring considered in this paper and* $1+J$ *be a normal subgroup of its unit group* R^*. *The following conditions hold:*

(i) The map $\psi : R_p \to End(1+J)$ *acting on each column of* R_p *is a surjective ring homomorphism.*

(ii) Let K *be the set of matrices* $A = \{(a_{ij}) : p^{e_i} \mid a_{ij} \forall i,j\}$. *This forms an ideal . The ideal* K *is the kernel of* ψ *and the endomorphism* $M = \psi(A) \cong R_p/Ker\psi$

(iii) The endomorphism $M = \psi(A)$ *is an automorphism if and only if* $A(modp) \in Gl_n(\mathbb{F}_p)$

3.1 Endomorphisms of $1+J$ and their properties for all the characteristics of R

We form R_p which is a set of matrices given by $R_p = \{(a_{ij}) : 1 \leq j \leq i \leq rk(1+J)\}$ for all the cases considered and determine the endomorphisms say $E_p = \psi(A)$ from that description

(a) If the characteristic of $R = p \neq 2$, $s = 1, t = 1$ and $\lambda = h-2$, then , $1+J = \mathbb{Z}_p^{r\lambda} \times \mathbb{Z}_{p^2}^r$. Clearly, $e_1 = ... = e_{r\lambda} = 1$ and $e_{(r\lambda)+1} = ... = e_{r(\lambda+1)} = 2$ so that R_p is given by

$$R_p = \begin{pmatrix} a_{11} & \cdots & a_{1(r\lambda)} & a_{1((r\lambda)+1)} & \cdots & a_{1(r(\lambda+1))} \\ a_{21} & \cdots & a_{2(r\lambda)} & a_{2((r\lambda)+1)} & \cdots & a_{2(r(\lambda+1))} \\ \vdots & & & & & \\ a_{(r\lambda)1} & \cdots & a_{(r\lambda)(r\lambda)} & a_{(r\lambda)((r\lambda)+1)} & \cdots & a_{(r\lambda)(r(\lambda+1)} \\ pa_{((r\lambda)+1)1} & \cdots & pa_{((r\lambda)+1)(r\lambda)} & a_{((r\lambda)+1)((r\lambda)+1)} & \cdots & a_{((r\lambda)+1)(r(\lambda+1))} \\ \vdots & & & & & \\ pa_{(r(\lambda+1))1} & \cdots & pa_{(r(\lambda+1))(r\lambda)} & a_{(r(\lambda+1))((r\lambda)+1)} & \cdots & a_{(r(\lambda+1))(r(\lambda+1))} \end{pmatrix}$$

Proposition 2. *If the characteristic of* $R = p$, $s = 1, t = 1$ *and* $p \neq 2$, $1+J = \mathbb{Z}_p^{r\lambda} \times \mathbb{Z}_{p^2}^r$, *then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq r(\lambda+1)\} = M_{r(\lambda+1)}(\mathbb{Z}_p)$

(ii) $rk(1+J) = r(\lambda+1)$

(iii) For $A \in R_p$ *and a surjective ring homomorphism*

$$\psi : R_p \to (\mathbb{Z}_p^{r\lambda} \times \mathbb{Z}_{p^2}^{r}), End(\mathbb{Z}_p^{r\lambda} \times \mathbb{Z}_{p^2}^{r}) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_p^{r\lambda} \times \mathbb{Z}_{p^2}^{r}) = GL_{(r(\lambda+1))}(\mathbb{F}_p)$

(b) If the characteristic of $R = p^2$, $s = h-1$, $t = 1$ and $\lambda = 0$, then $1 + J = \mathbb{Z}_p^{2r} \times \mathbb{Z}_{p^2}^{rs}$. Clearly $e_1 = ... = e_{2r} = 1$ and $e_{2r+1} = ... = e_{r(s+2)} = 2$ so that

$$R_p = \begin{pmatrix} a_{11} & \cdots & a_{1(2r)} & a_{1(2r+1)} & \cdots & a_{1(r(s+2))} \\ a_{21} & \cdots & a_{2(2r)} & a_{2((2r+1)} & \cdots & a_{2(r(s+2))} \\ \vdots & & & & & \\ a_{(2r)1} & \cdots & a_{(2r)(2r)} & a_{(2r)((2r)+1)} & \cdots & a_{(2r)(r(s+2)} \\ pa_{((2r)+1)1} & \cdots & pa_{((2r)+1)(2r)} & a_{((2r)+1)((2r)+1)} & \cdots & a_{((2r)+1)(r(s+2))} \\ \vdots & & & & & \\ pa_{(r(s+2))1} & \cdots & pa_{(r(s+2))(2r)} & a_{(r(s+2))((2r)+1)} & \cdots & a_{(r(s+2))(r(s+2))} \end{pmatrix}$$

Proposition 3. *If the characteristic of* $R = p^2$, $s = h-1$, $t = 1$ *and* $\lambda = 0$, $1 + J = \mathbb{Z}_p^{2r} \times \mathbb{Z}_{p^2}^{rs}$, *then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq r(s+2)\} = M_{r(s+2)}(\mathbb{Z}_p)$

(ii) $rk(1+J) = r(s+2)$

(iii) For $A \in R_p$ *and a surjective ring homomorphism*

$$\psi : R_p \to (\mathbb{Z}_p^{2r} \times \mathbb{Z}_{p^2}^{rs}), End(\mathbb{Z}_p^{2r} \times \mathbb{Z}_{p^2}^{rs}) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_p^{2r} \times \mathbb{Z}_{p^2}^{rs}) = GL_{r(s+2)}(\mathbb{F}_p)$

(c) If the characteristic of $R = p^3, s = h-1, t = 1$ and $\lambda = 0$

(I) when $p = 2$, $1 + J = \mathbb{Z}_2^{2+rs-r} \times \mathbb{Z}_4^{2r-1} \times \mathbb{Z}_8^{r}$. In this case $e_1 = 1 = ... = e_{2+rs-r}$, $e_{3+rs-r} = 2 = ... = e_{1+rs+r}$ and $e_{2+rs+r} = 3 = ... = e_{1+rs+2r}$. Therefore $R_p =$

$$\begin{pmatrix} a_{11} & \cdots & a_{1(\mu)} & \cdots & a_{1(v)} & \cdots & a_{1(k)} \\ \vdots & & & & & & \\ a_{(2+rs-r)1} & \cdots & a_{(2+rs-r)(\mu)} & \cdots & a_{(2+rs-r)(v)} & \cdots & a_{(2+rs-r)(k)} \\ 2a_{((2+rs-r)+1)1} & \cdots & 2a_{((2+rs-r)+1)(\mu)} & \cdots & 2a_{((2+rs-r)+1)(v)} & \cdots & a_{((2+rs-r)+1)(k)} \\ \vdots & & & & & & \\ 2a_{(1+rs+r)1} & \cdots & 2a_{(1+rs+r)(\mu)} & \cdots & 2a_{(1+rs+r)(v)} & \cdots & a_{(1+rs+r)(k)} \\ 4a_{((1+rs+r)+1)1} & \cdots & 4a_{((1+rs+r)+1)(\mu)} & \cdots & 2a_{((1+rs+r)+1)(v)} & \cdots & a_{((1+rs+r)+1)(k)} \\ \vdots & & & & & & \\ 4a_{(1+rs+2r)1} & \cdots & 4a_{(1+rs+2r)(\mu)} & \cdots & 2a_{(1+rs+2r)(v)} & \cdots & a_{(1+rs+2r)(k)} \end{pmatrix}$$

where $\mu = 2 + rs - r$, $v = 1 + rs + r$ and $k = 1 + rs + 2r = rk(1 + J)$

Proposition 4. *If the characteristic of $R = p^3, s = h - 1, t = 1$ and $\lambda = 0$, $p = 2$, $1 + J = \mathbb{Z}_2^{2+rs-r} \times \mathbb{Z}_4^{2r-1} \times \mathbb{Z}_8^r$ then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq 1 + rs + 2r\} = M_{(1+rs+2r)}(\mathbb{Z}_2)$

(ii) $rk(1 + J) = 1 + rs + 2r$

(iii) For $A \in R_p$ and a surjective ring homomorphism

$$\psi : R_p \to (\mathbb{Z}_2^{2+rs-r} \times \mathbb{Z}_4^{2r-1} \times \mathbb{Z}_8^r), End(\mathbb{Z}_2^{2+rs-r} \times \mathbb{Z}_4^{2r-1} \times \mathbb{Z}_8^r) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_2^{2+rs-r} \times \mathbb{Z}_4^{2r-1} \times \mathbb{Z}_8^r) = GL_{1+rs+2r}(\mathbb{F}_2)$

(II) when $p \neq 2$, $1 + J = \mathbb{Z}_{p^2}^{r(s+2)}$, then $e_1 = e_2 = ... = e_{r(s+2)} = 2$. Therefore,

$$R_p = \begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1(r+1)} & \cdots & a_{1(r(s+2))} \\ a_{21} & \cdots & a_{2r} & a_{2(r+1)} & \cdots & a_{2(r(s+2))} \\ \vdots & & & & & \\ a_{r1} & \cdots & a_{rr} & a_{r(r+1)} & \cdots & a_{r(r(s+2))} \\ a_{(r+1)1} & \cdots & a_{(r+1)r} & a_{(r+1)(r+1)} & \cdots & a_{(r+1)(r(s+2))} \\ \vdots & & & & & \\ a_{(r(s+2))1} & \cdots & a_{(r(s+2))r} & a_{(r(s+2))(r+1)} & \cdots & a_{(r(s+2))(r(s+2))} \end{pmatrix}$$

Proposition 5. *If the characteristic of $R = p^3, s = h - 1, t = 1$ and $\lambda = 0$, $p \neq 2$, $1 + J = \mathbb{Z}_{p^2}^{r(s+2)}$ then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq r(s+2)\} = M_{(r(s+2))}(\mathbb{Z}_p)$

(ii) $rk(1 + J) = r(s + 2)$

(iii) For $A \in R_p$ and a surjective ring homomorphism

$$\psi : R_p \to (\mathbb{Z}_{p^2}^{r(s+2)}), End(\mathbb{Z}_{p^2}^{r(s+2)}) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_{p^2}^{r(s+2)}) = GL_{r(s+2)}(\mathbb{F}_p)$

(d) If the characteristic of $R = p^4$, $s = h$, $t = 0$, $\lambda = 0$

(I) when $p = 2$ and $1 + J = \mathbb{Z}_2^{rs+1} \times \mathbb{Z}_{2^2}^1 \times \mathbb{Z}_{2^3}^{r-1}$, then $e_1 = ... = e_{rs+1} = 1$, $e_2 = e_{rs+2} = 2$ and finally $e_3 = e_{rs+3} = ... = e_{rs+r+1} = 3$, and it follows by definition that

$$R_p =$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1(1+rs)} & \cdots & a_{1(2+rs)} & \cdots & a_{1(1+rs+r)} \\ \vdots & & & & & & \\ a_{(1+rs)1} & \cdots & a_{(1+rs)(1+rs)} & \cdots & a_{(1+rs)(2+rs)} & \cdots & a_{(1+rs)(1+rs+r)} \\ \vdots & & & & & & \\ \vdots & & & & & & \\ 2a_{(2+rs)1} & \cdots & 2a_{(2+rs)(1+rs)} & \cdots & 2a_{(2+rs)(2+rs)} & \cdots & a_{(2+rs)(1+rs+r)} \\ 4a_{(3+rs)1} & \cdots & 4a_{(3+rs)(1+rs)} & \cdots & 2a_{(3+rs)(2+rs)} & \cdots & a_{(3+rs)(1+rs+r)} \\ \vdots & & & & & & \\ 4a_{(1+rs+r)1} & \cdots & 4a_{(1+rs+r)(1+rs)} & \cdots & 2a_{((1+rs+r)(2+rs)} & \cdots & a_{(1+rs+r)(1+rs+r)} \end{pmatrix}$$

Proposition 6. *If the characteristic of* $R = p^4$, $s = h$, $t = 0$, $\lambda = 0$ $p = 2$ *and* $1 + J = \mathbb{Z}_2^{rs+1} \times \mathbb{Z}_{2^2}^1 \times \mathbb{Z}_{2^3}^{r-1}$, *then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq 1 + rs + r\} = M_{(1+rs+r)}(\mathbb{Z}_2)$

(ii) $rk(1 + J) = 1 + rs + r$

(iii) For $A \in R_p$ *and a surjective ring homomorphism*

$$\psi : R_p \to (\mathbb{Z}_2^{rs+1} \times \mathbb{Z}_{2^2}^1 \times \mathbb{Z}_{2^3}^{r-1}), End(\mathbb{Z}_2^{rs+1} \times \mathbb{Z}_{2^2}^1 \times \mathbb{Z}_{2^3}^{r-1}) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_2^{rs+1} \times \mathbb{Z}_{2^2}^1 \times \mathbb{Z}_{2^3}^{r-1}) = GL_{1+rs+r}(\mathbb{F}_2)$

(II) when $p \neq 2$ and $1 + J = \mathbb{Z}_p^{rs} \times \mathbb{Z}_{p^3}^r$ then $e_1 = ... = e_{rs} = 1$ and $e_{rs+1} = ... = e_{r(s+1)} = 3$. Thus

$$R_p = \begin{pmatrix} a_{11} & \cdots & a_{1(rs)} & a_{1(rs+1)} & \cdots & a_{1(r(s+1))} \\ a_{21} & \cdots & a_{2(rs)} & a_{2(rs+1)} & \cdots & a_{2(r(s+1))} \\ \vdots & & & & & \\ a_{(rs)1} & \cdots & a_{(rs)(rs)} & a_{(rs)(rs+1)} & \cdots & a_{(rs)(r(s+1))} \\ p^2 a_{(rs+1)1} & \cdots & p^2 a_{(rs+1)(rs)} & a_{(rs+1)(rs+1)} & \cdots & a_{(rs+1)(r(s+1))} \\ \vdots & & & & & \\ p^2 a_{(r(s+1))1} & \cdots & p^2 a_{(r(s+1))(rs)} & a_{(r(s+1))(rs+1)} & \cdots & a_{(r(s+1))(r(s+1))} \end{pmatrix}$$

Proposition 7. *If the characteristic of* $R = p^4$, $s = h$, $t = 0$, $\lambda = 0$ $p \neq 2$ *and* $1 + J = \mathbb{Z}_p^{rs} \times \mathbb{Z}_{p^3}^r$, *then*

(i) $R_p = \{(a_{ij}) : p^{e_i - e_j} \mid a_{ij}, \forall i, j : 1 \leq j \leq i \leq r(s+1)\} = M_{(r(s+1))}(\mathbb{Z}_p)$

(ii) $rk(1 + J) = r(s + 1)$

(iii) For $A \in R_p$ *and a surjective ring homomorphism*

$$\psi : R_p \to (\mathbb{Z}_p^{rs} \times \mathbb{Z}_{p^3}^r), End(\mathbb{Z}_p^{rs} \times \mathbb{Z}_{p^3}^r) = \psi(A)$$

(iv) $Aut(\mathbb{Z}_p^{rs} \times \mathbb{Z}_{p^3}^r) = GL_{r(s+1)}(\mathbb{F}_p)$

At this point, we give the structure of $Aut(R^*)$:

Theorem 2. *The structure of the automorphisms* $Aut(R^*)$ *of the unit group* R^* *of the commutative completely primary finite ring* R *of characteristic* p, p^2, p^3, p^4 *with maximal ideal* J *such that* $J^4 = (0)$ *and* $J^3 \neq (0)$, *with the invariants* p, r, s, t, h *and* λ *where* $p \in J$, *is characterized as follows:*

(i) If the $charR = p$ $s = 1, t = 1$ *and* $\lambda = h - 2$, *then,* $\forall p$

$$Aut(R^*) \cong (\mathbb{Z}_{p^r-1})^* \times GL_{(r(\lambda+1))}(\mathbb{F}_p).$$

(ii) If the $charR = p^2$ $s = h - 1$, $t = 1$ *and* $\lambda = 0$ *then,* $\forall p$

$$Aut(R^*) \cong (\mathbb{Z}_{p^r-1})^* \times GL_{(r(s+2))}(\mathbb{F}_p)$$

(iii) If the $charR = p^3$ $s = h - 1, t = 1$ *and* $\lambda = 0$ *then,*

$$Aut(R^*) \cong \begin{cases} (\mathbb{Z}_{2^r-1})^* \times GL_{(r(1+rs+2r))}(\mathbb{F}_2), & \text{if } p = 2; \\ (\mathbb{Z}_{p^r-1})^* \times GL_{(r(s+2))}(\mathbb{F}_p), & \text{if } p \neq 2. \end{cases}$$

(iv) If the $charR = p^4$ $s = h$, $t = 0$, $\lambda = 0$ *then,*

$$Aut(R^*) \cong \begin{cases} (\mathbb{Z}_{2^r-1})^* \times GL_{(1+rs+r)}(\mathbb{F}_2), & \text{if } p = 2; \\ (\mathbb{Z}_{p^r-1})^* \times GL_{(r(s+1))}(\mathbb{F}_p), & \text{if } p \neq 2. \end{cases}$$

4 Counting the Automorphisms of (1+J)

It can be noted that, the structure of $Aut(1 + J)$ is a general linear group $GL_{rk(1+J)}(\mathbb{F}_p)$. So, we need to count them:

Remark 1. Let $\mathbb{F}$ be a field. Then, we define a general linear group $GL_{rk(1+J)}(\mathbb{F})$ as the group of invertible $rk(1+J) \times rk(1+J)$ matrices with entries in $\mathbb{F}$ under matrix multiplication.

Intuitively, $GL_{rk(1+J)}(\mathbb{F})$ is a group because: matrix multiplication is associative, the identity element is $I_{rk(1+J)}$; the $rk(1+J)\times rk(1+J)$ matrix with $1'$s along the main diagonal and zeros elsewhere. If $a \in \mathbb{F}$, $a \neq 0$, then $a \cdot I_{rk(1+J)}$ is an invertible $rk(1+J) \times rk(1+J)$ matrix with inverse $a^{-1}I_{rk(1+J)\times rk(1+J)}$. In fact, the set of all such matrices forms a subgroup of $GL_{rk(1+J)}(\mathbb{F})$ that is isomorphic to $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$

Since $\mathbb{F}_p$ is a finite field in all the cases, it is immediate that $GL_{rk(1+J)}(\mathbb{F}_p)$ has only finitely many elements. Now, suppose $rk(1+J) = 1$, then $GL_{rk(1+J)}(\mathbb{F}_p) \cong \mathbb{F}_p^\times$ has $p-1$ elements.

Let $rk(1+J) = 2$, and let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for M to be invertible, it is necessary and sufficient that $ad \neq bc$. If a, b, c and d are all nonzero, then, we can fix a, b, c arbitrarily and d can be anything but not $a^{-1}bc$. This gives us $(p^3-1)(p-2)$ matrices. If exactly one of the entries is 0, then the other three can be anything nonzero for a total of $4(p-1)^3$ matrices. Finally, if exactly two entries are 0, then, these entries must be opposite each other for the matrix to be invertible and the other two entries can be anything nonzero for a total of $2(p-1)^2$ matrices. So altogether we have:

$$\begin{aligned} &(p^3-1)(p-2) + 4(p-1)^3 + 2(p-1)^2 \\ &= (p-1)^2((p-1)(p-2) + 4(p-1) + 2) \\ &= (p-1)^2(p^2+p) = (p^2-1)(p^2-p) \end{aligned}$$

invertible matrices

Evidently, when calculating the size of $GL_{rk(1+J)}(\mathbb{F})$ by directly calculating the determinant, then, determining what values of the entries make the determinant nonzero is quite a tedious exercise and hence error prone. But one of the basic properties of determinants is that the determinant of a matrix is nonzero if and only if the rows of the matrix are linearly independent. Thus we have at once:

Proposition 8. *Let $rk(1+J) = n$ and $1 + J = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n}$. The number of elements in $GL_n(\mathbb{F}_p)$ is $\prod_{k=0}^{n-1}(p^n - p^k)$*

Proof. Notice that $e_1 = e_2 = ...e_{rk(1+J)} = 1$. We count the number of $n \times n$ matrices $A \in R_p$ whose rows are linearly independent. This is done by building the matrix A from scratch. The first row can be anything other than the zero row, so, there are $p^n - 1$ possibilities. The second row must be linearly independent from the first row. Since there are p multiples of the first row, there are $p^n - p$ possibilities for the second row. Generally, the i^{th} row must be linearly independent from the first $i-1$ rows. There are p^{i-1} linear

combinations of the first $i-1$ rows, so, there are $p^n - p^{i-1}$ possibilities for the i^{th} row. Once we build the entire matrix this way, we know that the rows are all linearly independent by choice. Also, we can build any $n \times n$ matrix whose rows are linearly independent in this fashion. Thus there are

$$(p^n-1)(p^n-p)\cdots(p^n-p^{n-1}) = \prod_{k=0}^{n-1}(p^n-p^k)$$

matrices □

In order to exhaust this process of counting $Aut(1+J)$, we need to find all the elements of $GL_{rk(1+J)}(\mathbb{F}_p)$ that can be extended to a matrix in $End(1+J)$ and calculate the distinct ways of extending such an element to an endomorphism.

We define the following numbers:

$$\alpha_k = max\{m : e_m = e_k\}, \beta_k = min\{m : e_m = e_k\}$$

Since $e_m = e_k$ for $m = k$, we have the two inequalities $\alpha_k \geq k$ and $\beta_k \leq k$

Note that $\beta_1 = \beta_2 = ... = \beta_{\alpha_1}$, so we have $\beta_1 = \cdots = \beta_{\alpha_1} \leq \beta_{\alpha_1+1}$

When $e_i = e_j = \cdots = e_{rk(1+J)}$, for all i, j then it follows that $\alpha_k = \beta_k$

Suppose the e_i are different, we can introduce the numbers $e_i^{'}, c_i, d_i$ as follows. Define the set of distinct numbers $\{e_i^{'}\}$ such that $\{e_i^{'}\} = \{e_j\}$ and $e_1^{'} < e_2^{'} < \cdots$

Let $l \in \mathbb{N}$ be the size of $\{e_i^{'}\}$. So, $e_1^{'} = e_1$, $e_2^{'} = e_{\alpha_1+1}, \cdots, e_l^{'} = e_n$. Now define

$$d_i = max\{m : e_m = e_i^{'}\}, c_i = min\{m : e_m = e_i^{'}\}$$

Note that $c_1 = 1$, and $d_l = rk(1+J)$. Also, for convenience define $c_{l+1} = (rk(1+J))+1$

Now, for both of the considerations, the number of matrices say $A \in R_p$ that are invertible modulo p are upper block triangular matrices which may be expressed in the following three forms

$$A = \begin{pmatrix} m_{11} & & & & & & & & * \\ \vdots & & & & & & & & \\ m_{d_1 1} & \cdots & m_{d_1 d_1} & & & & & & \\ & & & m_{c_2 c_2} & & & & & \\ & & & \vdots & & & & & \\ & & & m_{d_2 c_2} & \cdots & m_{d_2 d_2} & & & \\ & & & & & & \ddots & & \\ & & & & & & & m_{c_1 c_1} & \\ & & & & & & & \vdots & \\ 0 & & & & & & & m_{d_l c_l} & \cdots & m_{d_l c_l} \end{pmatrix}$$

or

$$A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1(rk(1+J))} \\ \vdots & & & \\ m_{\alpha_1 1} & & & \\ & m_{\alpha_2 2} & & \\ & & \ddots & \\ 0 & & & m_{\alpha_{(rk(1+J))}(rk(1+J))} \end{pmatrix}$$

$$= \begin{pmatrix} m_{1\beta_1} & & & & & \\ & m_{2\beta_2} & & & & \\ & & \ddots & & & \\ 0 & & & m_{(rk(1+J))\beta_{(rk(1+J))}} & \cdots & m_{(rk(1+J))(rk(1+J))} \end{pmatrix}$$

The number of such A is $\prod_{k=1}^{rk(1+J)}(p^{\alpha_k} - p^{k-1})$ since we require linearly independent columns. So, the first step of calculating $\mid Aut(1+J) \mid$ is done.

The second half of the computation is to count the number of extensions of A to $Aut(1+J)$. To extend each entry m_{ij} from $m_{ij} \in \mathbb{Z}/p\mathbb{Z}$ to $a_{ij} \in p^{e_i - e_j}\mathbb{Z}/p^{e_i}\mathbb{Z}$ if $e_i > e_j$, or $a_{ij} \in \mathbb{Z}/p^{e_i}\mathbb{Z}$ if $e_i \leq e_j$, such that $a_{ij} \equiv m_{ij}(\mathrm{mod})p$, we have p^{e_j} ways to do so for the necessary zeros (that is , when $e_i > e_j$) as any element of $p^{e_i - e_j}\mathbb{Z}/p^{e_i}\mathbb{Z}$ works.

Similarly, there are $p^{e_i - 1}$ ways for the not necessarily zero entries (that is , when $e_i \leq e_j$) as any element of $p\mathbb{Z}/p^{e_i}\mathbb{Z}$ will do.

Finally, as a result of this process, we have the result below:

Theorem 3. *The number of the automorphisms $Aut(R^*)$ of the unit group R^* of the commutative completely primary finite ring R of characteristic p, p^2, p^3, p^4 with maximal ideal J such that $J^4 = (0)$ and $J^3 \neq (0)$, with the invariants p, r, s, t, h and λ where $p \in J$, is characterized as follows:*

(i) If the $charR = p$ $s = 1, t = 1$ and $\lambda = h - 2$, then,

$$\mid Aut(R^*) \mid = \varphi(p^r - 1) \cdot \prod_{k=1}^{r(\lambda+1)} (p^{\alpha_k} - 2^{k-1}) \prod_{j=1}^{r(\lambda+1)} (p^{e_j})^{r(\lambda+1)-\alpha_j} \prod_{i=1}^{r(\lambda+1)} (p^{e_i - 1})^{r(\lambda+1)-\beta_i+1}$$

(ii) If the $charR = p^2$ $s = h - 1$, $t = 1$ and $\lambda = 0$ then, $\forall p$

$$\mid Aut(R^*) \mid = \varphi(p^r - 1) \cdot \prod_{k=1}^{r(s+2)} (p^{\alpha_k} - 2^{k-1}) \prod_{j=1}^{r(s+2)} (p^{e_j})^{r(s+2)-\alpha_j} \prod_{i=1}^{r(s+2)} (p^{e_i - 1})^{r(s+2)-\beta_i+1}$$

(iii) If the $charR = p^3$ $s = h - 1, t = 1$ *and* $\lambda = 0$ *then,* $\mid Aut(R^*) \mid =$

$$\begin{cases} \varphi(2^r-1)\cdot\prod_{k=1}^{1+rs+2r}(2^{\alpha_k}-2^{k-1})\prod_{j=1}^{1+rs+2r}(2^{e_j})^{(1+rs+2r)-\alpha_j}\prod_{i=1}^{w}(2^{e_i-1})^{w-\beta_i+1}, & \text{if } p=2; \\ \varphi(p^r-1)\cdot\prod_{k=1}^{r(s+2)}(p^{\alpha_k}-2^{k-1}), & \text{if } p\neq 2. \end{cases}$$

where $w = (1 + rs + 2r)$

(iv) If the $charR = p^4$ $s = h$, $t = 0$, $\lambda = 0$ *then,* $\mid Aut(R^*) \mid =$

$$\begin{cases} \varphi(2^r-1)\cdot\prod_{k=1}^{1+rs+r}(2^{\alpha_k}-2^{k-1})\prod_{j=1}^{1+rs+r}(2^{e_j})^{(1+rs+r)-\alpha_j}\prod_{i=1}^{1+rs+r}(2^{e_i-1})^{w-\beta_i+1}, & \text{if } p=2; \\ \varphi(p^r-1)\cdot\prod_{k=1}^{r(s+1)}(p^{\alpha_k}-2^{k-1})\prod_{j=1}^{r(s+1)}(p^{e_j})^{r(s+1)-\alpha_j}\prod_{i=1}^{r(s+1)}(p^{e_i-1})^{r(s+1)-\beta_i+1}, & \text{if } p\neq 2. \end{cases}$$

where $w = (1 + rs + r)$

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