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Abstract

Let R_o be a Galois ring and U is a finitely generated R_o- module. Consider an idealization of U expressed as $R = R_o \oplus U$ endowed with a suitable multiplication. We explore the structure of R through its group of units $R^\times$ and the graph of its zero divisors $\Gamma(R)$. The study involves an investigation on the overarching interplay between the ring theoretical properties of R, the group theoretic properties of $R^\times$ and the graph theoretic properties of $\Gamma(R)$. Since R is a finite ring with identity, the convention that each element of R is either a unit or a zero divisor has been extensively used to drive the concept of classification of the elements of R. The units of R have been classified, the automorphisms of R have been determined and the zero divisors of R have been characterized.

Keywords: Galois ring; idealization; units; zero, divisors.

1 Introduction

R denotes a ring with identity, while $R^\times$ denotes its corresponding group of units. Unless otherwise stated $Z(R)$ denotes the set of zero divisors of R while $\mathcal{J}(R)$ denotes the Jacobson radical of R. The coefficient (Galois) subring of R of order p^{kr} and characteristic p^k is denoted by $R_o = GR(p^{kr}, p^k)$. The quotient $R_o/\mathcal{J}(R_o)$ shall be denoted by $\mathbb{F}_o$ to represent, the Galois Field of order p^r, for a prime integer p and positive integer r. For each $x \in R$, $o(x)$ denotes the order of x. For every finite subset X of R, $\mid X \mid$ denotes the cardinality of X and $N(X)$ denotes the neighbourhood of X. The automorphism group of R is denoted by $\mathrm{Aut}(R)$ while

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char$R$ denotes the characteristic of R. The zero divisor graph of R is denoted by $\Gamma(R)$. The diameter of the graph is denoted by $diam(\Gamma(R))$ while the girth and the binding number of the graph are denoted by $gr(\Gamma(R))$ and $b(\Gamma(R))$ respectively.

2 The Construction of R

Let R_o be a Galois ring of the form $\mathrm{GR}(p^{kr}, p^k)$. For each $i = 1, ..., h$, let $u_i \in Z(R)$ such that U is an $h-$ dimensional R_o module generated by $\{u_1, ..., u_h\}$ so that $R = R_o \oplus U = R_o \oplus R_o u_1 \oplus ... \oplus R_o u_h$ is an additive group. On this group, define multiplication as follows:

$$(r_o, \sum_{i=1}^{h} r_i u_i)(r_o^{'}, \sum_{i=1}^{h} r_i^{'} u_i) = (r_o r_o^{'}, \sum_{i=1}^{h} [(r_o + pR_o) r_i^{'} + r_i (r_o^{'} + pR_o)^{\sigma_i}] u_i)$$

where $\sigma_i \in \mathrm{Aut}(R_o)$.

From the construction, it is easy to note the following:

1. The multiplication turns R into a ring such that $Z(R)$ coincides with the Jacobson radical $\mathcal{J}(R)$ which equals to $pR_o \oplus U$.
2. The multiplication turns R into a commutative ring iff $\sigma_i = \mathrm{id}_{R_o}$.
3. If R is commutative, then it is an R_o module U idealization with identity $(1, 0, ...0)$.
4. $\mid Z(R) \mid = p^{(n-1)r}$ and $\mid R \mid = p^{nr}$ for some prime integer p and positive integers n, r.
5. With respect to the multiplication,
$$(\mathcal{J}(R))^{n-1} \neq (0); (\mathcal{J}(R))^n = (0).$$
6. Char $R = p^k$ for some integer k with $1 \leq k \leq n$.
7. In terms of the relations between elements of R,
(i) If $k = 1, 2$ then $pu_i = u_i u_j = u_j u_i = 0, u_i r_o = (r_o)^{\sigma_i} u_i$
(ii) If $k \geq 3$, then $p^{k-1} u_i = 0, u_i u_j = p^2 \gamma_{ij}, u_i^k = u_i^{k-1} u_j, u_i u_j^{k-1} = 0, u_i r_o = (r_o)^{\sigma_i} u_i$, where $r_o, \gamma_{ij} \in R_o$ and $1 \leq i, j \leq h$ and σ_i is the automorphism associated with u_i.

Using the restriction $u_i|_U$ so that $pu_i = u_iu_j = 0$, we obtain an idealization in which the order of each generator u_i, $o(u_i) = p$. In the sequel, we consider a class of idealizations satisfying this condition.

The following result is useful.

Theorem 1. *Let R_o =$GR(p^{kr}, p^k)$, $k \geq 3$. Then the idealization $R = R_o \oplus U$ is completely primary and of characteristic p^k satisfying:*
(i)$\mathcal{J}(R) = pR_o \oplus U$
(ii) $(\mathcal{J}(R))^{k-1} = p^{k-1}R_o$
(iii) $(\mathcal{J}(R))^k = (0)$.

In the sequel, a ring that satisfies the above three properties with the restriction $u_i|_U$ so that $pu_i = u_iu_j = 0$, shall be called, a ring with property P.

3 Units of Idealization Rings

We start by investigating the structure of the unit groups of idealization rings constructed in this paper. Similar studies can be obtained from [1], [2], [3], [4], [5], [6], [7].

Lemma 1. *Let R be an R_o- module idealization with $u_i|_U$. Then $R^\times$ is cyclic iff $1 + \mathcal{J}(R)$ is cyclic. Moreover,*

$$R^\times =< b > .(1 + \mathcal{J}(R)) \cong < b > \times(1 + \mathcal{J}(R)),$$

a direct product of the p-group $1 + \mathcal{J}(R)$ by the cyclic subgroup $< b >$, where the order, $o(b) = p^r - 1$

The following result due to Raghavendran[8] on the structure of the units of the Galois ring $R_o = GR(p^{kr}, p^k)$ shall be useful.

Theorem 2. *Let $R_o = GR(p^{kr}, p^k)$ be a Galois ring. Then $U(R_o)$ is a direct product of the cyclic group of order $p^r - 1$ by the group $1 + pR_o$ of order $p^{(k-1)r}$ whose structure is described as follows:*
(a)If (i) p is odd,or (ii) $p = 2$ and $k \leq 2$ then $1 + pR_o$ is the direct product of r cyclic groups each of order p^{k-1}.
(b) When $p = 2$ and $k \geq 3$, the group $1 + pR_0$ is the direct product of a cyclic group of order 2, a cyclic group of order 2^{k-2} and $r - 1$ cyclic groups each of order 2^{k-1}.

To completely classify $R^\times$, we need to determine the structure of $1 + \mathcal{J}(R)$. The abelianity of $R^\times$ implies that $1 + \mathcal{J}(R)$ is a normal subgroup of $R^\times$.

Let R be an R_o- module idealization with $u_i|_U$, then $1+\mathcal{J}(R)$ is an abelian $p-$ subgroup of $R^\times$ with a filtration

$$1+\mathcal{J}(R) \supset 1+(\mathcal{J}(R))^2 \supset ... \supset 1+(\mathcal{J}(R))^k = \{1\}$$

with filtration quotients

$$1+\mathcal{J}(R)/1+(\mathcal{J}(R))^2, 1+(\mathcal{J}(R))^2/1+(\mathcal{J}(R))^3, ..., 1+(\mathcal{J}(R))^{k-1}/1+(\mathcal{J}(R))^k = 1+(\mathcal{J}(R))^{k-1}$$

isomorphic to the additive groups (R_o- vector spaces)

$$\mathcal{J}(R)/(\mathcal{J}(R))^2, (\mathcal{J}(R))^2/(\mathcal{J}(R))^3, ..., (\mathcal{J}(R))^{k-1}/(\mathcal{J}(R))^k$$

respectively. The following results are important in determining the structure of $1+\mathcal{J}(R)$

Lemma 2. *Let p be prime integer. Then $1+pR_o$ is a subgroup of $1+\mathcal{J}(R)$.*

Proof. If $\text{char}R = p$, then $1+pR_o$ is a trivial subgroup of $1+\mathcal{J}(R)$. Next, we prove the result when $\text{char}R = p^2$ since the subsequent results easily follow from this. Now, each element of $1+pR_o$ is of the form $1+pr_o$, $r_o \in R_o$, and for any two elements $1+pr_o$ and $1+pr_o^{'}$, $(1+pr_o^{'})^{-1} = 1-pr_o^{'}$ and

$$(1+pr_o)(1+pr_o^{'})^{-1} = (1+pr_o)(1-pr_o^{'}) = 1+p(r_o-r_o^{'})$$

which is clearly an element of $1+pR_o$ since $p^2 r_o r_o^{'} = 0$. □

Proposition 1. *For each $1 \le i \le h$, $1+\sum_{i=1}^{h} \oplus R_o u_i$ is a subgroup of $1+\mathcal{J}(R)$*

Proof. For any two elements $1+\sum a_i u_i$ and $1+\sum b_i u_i$ in $1+\sum_{i=1}^{h} \oplus R_o u_i$, it follows that

$$\begin{aligned}(1+\sum a_i u_i)(1+\sum b_i u_i)^{-1} &= (1+\sum a_i u_i)(1-\sum b_i u_i) \\ &= 1+\sum (a_i - b_i)u_i\end{aligned}$$

an element of $1+\sum_{i=1}^{h} \oplus R_o u_i$ since $u_i^2 = 0$. □

Corollary 3. *$1+ann(\mathcal{J}(R))$ is a subgroup of $1+\mathcal{J}(R)$.*

Proof. Easily follows from the fact that $ann(\mathcal{J}(R)) = p^{k-1}R_o \oplus \sum_{i=1}^{h} R_o u_i$ □

Proposition 2. *The group $1+\mathcal{J}(R)$ is a direct product of the subgroups $1+pR_o$ by $1+\sum \oplus R_o u_i$*

Proof. Since $ann(\mathcal{J}(R)) \supseteq \sum_{i=1}^{h} \oplus R_o u_i$, it easily follows that

$$\begin{aligned}(1+pR_o) \times (1+\sum_{i=1}^{h} \oplus R_o u_i) &= 1+pR_o \oplus \sum_{i=1}^{h} R_o u_i \\ &= 1+\mathcal{J}(R)\end{aligned}$$

□

Proposition 3. *Let R be an R_o- module idealization. If $1 \leq i \leq h$, and $charR = p$, then $1 + \mathcal{J}(R) \cong (\mathbb{Z}_p^r)^h$*

Proof. Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$ form a basis for R_o/pR_o regarded as a vector space over its prime subfield $GF(p)$. For every $\nu = 1, ..., r$

$$(1 + \epsilon_\nu u_1)^p = 1, (1 + \sum_{i=1}^{2} \epsilon_\nu u_i)^p = 1, ..., (1 + \sum_{i=1}^{h} \epsilon_\nu u_i)^p = 1.$$

For positive integers $\beta_{1\nu}, ..., \beta_{h\nu}$ with $\beta_{i\nu} \leq p (1 \leq i \leq h, 1 \leq \nu \leq r)$, we notice that the equation $\prod_{\nu=1}^{r}\{(1 + \epsilon_\nu u_1)^{\beta_{1\nu}}\}.\prod_{\nu=1}^{r}\{(1 + \sum_{i=1}^{2} \epsilon_\nu u_i)^{\beta_{2\nu}}\}...\prod_{\nu=1}^{r}\{(1 + \sum_{i=1}^{h} \epsilon_\nu u_i)^{\beta_{h\nu}}\} = \{1\}$ will imply $\beta_{i\nu} = p$ for every $\nu = 1, ..., r$ and $1 \leq i \leq h$. If we set

$$S_{1\nu} = \{(1 + \epsilon_\nu u_1)^{\beta_1} \mid \beta_1 = 1, ..., p\}$$

$$S_{2\nu} = \{(1 + \sum_{i=1}^{2} \epsilon_\nu u_i)^{\beta_2} \mid \beta_2 = 1, ..., p\}$$

$$\vdots$$

$$S_{h\nu} = \{(1 + \sum_{i=1}^{h} \epsilon_\nu u_i)^{\beta_h} \mid \beta_h = 1, ..., p\}$$

we see that the hr groups $S_{1\nu}, ..., S_{h\nu}$ are cyclic subgroups of the group $1 + \mathcal{J}(R)$ and they are each of order p. Since the product of the orders of the hr subgroups equals to p^{hr} and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the hr subgroups is direct and exhausts $1 + \mathcal{J}(R)$. □

Example 1. *For a prime integer p, let $R = \mathbb{Z}_p \oplus \mathbb{Z}_p u_1 \oplus ... \oplus \mathbb{Z}_p u_h$. In this case, $R_o = \mathbb{Z}_p$, $r = k = 1$ and using the multiplication in Section 1, $\mathcal{J}(R) = \mathbb{Z}_p u_1 \oplus ... \oplus \mathbb{Z}_p u_h$. Clearly, from Lemma 1, $R^* \cong \mathbb{Z}_{p-1} \times (1 + \mathcal{J}(R))$ where $1 + \mathcal{J}(R) = 1 + \mathbb{Z}_p u_1 \oplus ... \oplus \mathbb{Z}_p u_h$. Now, the generators of $1 + \mathcal{J}(R)$ are*

$$1 + r_1 u_1, 1 + r_1 u_1 + r_2 u_2, ..., 1 + r_1 u_1 + r_2 u_2 + ... + r_h u_h$$

each of order p, $r_i \in \mathbb{Z}_p$, $1 \leq i \leq h$ and $1 + J(R) =< 1 + r_1 u_1 > \times < 1 + r_1 u_1 + r_2 u_2 > \times ... \times < 1 + r_1 u_1 + r_2 u_2 + ... + r_h u_h > \cong \underbrace{\mathbb{Z}_p \times ... \times \mathbb{Z}_p}_{h}$ so that

$$R^* \cong \mathbb{Z}_{p-1} \times \underbrace{\mathbb{Z}_p \times ... \times \mathbb{Z}_p}_{h}$$

Proposition 4. *Let R be an R_o- module idealization with $u_i|_U$. If $1 \leq i \leq h$, and $charR = p^2$, then $1 + \mathcal{J}(R) \cong \mathbb{Z}_p^r \times (\mathbb{Z}_p^r)^h$*

Proof. Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$ form a basis for R_o/pR_o regarded as a vector space over its prime subfield $GF(p)$. For every $\epsilon = 1, ..., r$ $(1+p\epsilon_\nu)^p = 1, (1+\epsilon_\nu u_1)^p = 1, (1+\sum_{i=1}^{2} \epsilon_\nu u_i)^p = 1, ..., (1+\sum_{i=1}^{h} \epsilon_\nu u_i)^p = 1$. For positive integers $\alpha_\nu, \beta_{1\nu}, ..., \beta_{h\nu}$ with $\alpha_\nu \leq p, \beta_{i\nu} \leq p (1 \leq i \leq h, 1 \leq \nu \leq r)$, we notice that the equation $\prod_{\nu=1}^{r}\{(1+p\epsilon_\nu)^{\alpha_\nu}\}.\prod_{\nu=1}^{r}\{(1+\epsilon_\nu u_1)^{\beta_{1\nu}}\}.\prod_{\nu=1}^{r}\{(1+\sum_{i=1}^{2} \epsilon_\nu u_i)^{\beta_{2\nu}}\}...\prod_{\nu=1}^{r}\{(1+\sum_{i=1}^{h} \epsilon_\nu u_i)^{\beta_{h\nu}}\} = \{1\}$ will imply $\alpha_\nu = \beta_{i\nu} = p$ for every $\nu = 1, ..., r$ and $1 \leq i \leq h$. Setting

$$T_\nu = \{(1+p\epsilon_\nu)^\alpha \mid \alpha = 1, ..., p\}$$

$$S_{1\nu} = \{(1+\epsilon_\nu u_1)^{\beta_1} \mid \beta_1 = 1, ..., p\}$$

$$S_{2\nu} = \{(1+\sum_{i=1}^{2} \epsilon_\nu u_i)^{\beta_2} \mid \beta_2 = 1, ..., p\}$$

$$\vdots$$

$$S_{h\nu} = \{(1+\sum_{i=1}^{h} \epsilon_\nu u_i)^{\beta_h} \mid \beta_h = 1, ..., p\}$$

we see that the $(h+1)r$ subgroups $T_\nu, S_{1\nu}, ..., S_{h\nu}$ are cyclic. It is easy to verify that the product of the $(h+1)r$ subgroups $T_\nu, S_{1\nu}, ..., S_{h\nu}$ is direct and exhausts $1 + \mathcal{J}(R)$. □

Example 2. *For a prime integer p, let* $R = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}u_1 \oplus ... \oplus \mathbb{Z}_{p^2}u_h$. *In this case,* $R_o = \mathbb{Z}_{p^2}$, $r = 1, k = 2$ *and using the multiplication in Section 1,* $\mathcal{J}(R) = p\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}u_1 \oplus ... \oplus \mathbb{Z}_{p^2}u_h$. *Clearly, from Lemma 1,* $R^* \cong \mathbb{Z}_{p-1} \times (1+\mathcal{J}(R))$ *where* $1+J(R) = 1+p\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}u_1 \oplus ... \oplus \mathbb{Z}_{p^2}u_h$. *Now, the generators of* $1+\mathcal{J}(R)$ *are* $1+pr_o, 1+r_1u_1, 1+r_1u_1+r_2u_2, ..., 1+r_1u_1+r_2u_2+...+r_hu_h$ *each of order p where* $r_o \in \mathbb{Z}_{p^2}, r_i \in \mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2}$, $1 \leq i \leq h$ *and* $1 + J(R) =< 1 + pr_o > \times < 1 + r_1u_1 > \times < 1 + r_1u_1 + r_2u_2 > \times ... \times < 1 + r_1u_1 + r_2u_2 + ... + r_hu_h > \cong \underbrace{\mathbb{Z}_p \times ... \times \mathbb{Z}_p}_{h+1}$ *so that*

$$R^* \cong \mathbb{Z}_{p-1} \times \underbrace{\mathbb{Z}_p \times ... \times \mathbb{Z}_p}_{h+1}$$

Proposition 5. *Let R be ring with property P, then*

$$1 + J(R) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} \times \mathbb{Z}_{2^{k-1}}^{r-1} \times (\mathbb{Z}_2^r)^h & if\ p = 2 \\ \mathbb{Z}_{p^{k-1}}^r \times (\mathbb{Z}_p^r)^h & if\ p \neq 2 \end{cases}$$

Proof. Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$ form a basis for R_o/pR_o regarded as a vector space over its prime subfield $GF(p)$. Since the two cases do not overlap, they are considered separately.

Case 1: $p = 2$.
For each $w \in 1 + pR_o$. Setting

$$H = \{(-1 + 2^{k-1}\epsilon_1)^{\alpha} \mid \alpha = 1, 2\}$$

$$Q = \{(1 + 4t)^{\beta} \mid \beta = 1, ..., 2^{n-2}\}$$

$$T_\nu = \{(1 + 2\epsilon_\nu)^{\kappa} \mid \kappa = 1, ..., 2^{k-1}\}, \nu = 2, ..., r$$

$$S_{1\nu} = \{(1 + \epsilon_\nu u_i)^{\tau_1} \mid \tau_1 = 1, 2\}, \nu = 1, ..., r$$

$$S_{2\nu} = \{(1 + \sum_{i=1}^{2} \epsilon_\nu u_i)^{\tau_2} \mid \tau_2 = 1, 2\}, \nu = 1, ..., r$$

$$\vdots$$

$$S_{h\nu} = \{(1 + \sum_{i=1}^{h} \epsilon_\nu u_i)^{\tau_h} \mid \tau_h = 1, 2\}, \nu = 1, ..., r$$

we see that the $(h + 1)r + 1$ subgroups $H, Q, T_\nu, S_{1\nu}, ..., S_{h\nu}$ are cyclic and their direct product exhausts $1 + \mathcal{J}(R)$.

Case 2: $p \neq 2$. Suppose $\nu = 1, ..., r$ and setting

$$T_\nu = \{(1 + p\epsilon_\nu)^{\alpha} \mid \alpha = 1, ..., p^{k-1}\},$$

$$S_{1\nu} = \{(1 + \epsilon_\nu u_1)^{\beta_1} \mid \beta_1 = 1, ..., p\},$$

$$S_{2\nu} = \{(1 + \sum_{i=1}^{2} \epsilon_\nu u_i)^{\beta_2} \mid \beta_2 = 1, ..., p\},$$

$$\vdots$$

$$S_{h\nu} = \{(1 + \sum_{i=1}^{h} \epsilon_\nu u_i)^{\beta_h} \mid \tau_h = 1, ..., p\},$$

we see that the $(h+1)r$ subgroups $T_\nu, S_{1\nu}, ...S_{h\nu}$ are cyclic and their direct product exhausts $1 + \mathcal{J}(R)$. □

The results proved in this section can be summarized by the following Theorem.

Theorem 4. *The unit group $R^{\times}$ of the idealization ring R, with invariants p (prime integer), $p \in \mathcal{J}(R)$, $r \geq 1$ and $h \geq 1$, is a direct product of cyclic groups as follows:*

$$R^{\times} \cong \begin{cases} \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h & if\ charR = p \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_p^r)^h & if\ charR = p^2 \\ \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} \times \mathbb{Z}_{2^{k-1}}^{r-1} \times (\mathbb{Z}_2^r)^h & if\ charR = p^k, p = 2, k \geq 3 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^{k-1}}^r \times (\mathbb{Z}_p^r)^h & if\ charR = p^k, p \neq 2, k \geq 3 \end{cases}$$

4 Automorphism Groups of Idealization Rings

Let $R = R_o \oplus R_o u_1 \oplus ... \oplus R_o u_h$, where $u_i \in \mathcal{J}(R)$ and that

$$\mathcal{J}(R) = pR_o \oplus R_o u_1 \oplus ... \oplus R_o u_h.$$

Since every element of R_o is expressible in the form $\alpha_o + \alpha_1 p + ... + \alpha_{k-1} p^{k-1}$ with $\alpha_o, \alpha_1, ..., \alpha_{k-1} \in \mathbb{F}_o$, it follows that every element of R is expressible in the form

$$\alpha_o + \alpha_1 p + ... + \alpha_{k-1} p^{k-1} + \lambda_1 u_1 + ... + \lambda_h u_h,$$

where $\lambda_i \in \mathbb{F}_o, 1 \leq i \leq h$. Clearly, the non- zero elements of

$$\{1, p, p^2, ..., p^{k-1}, u_1, ..., u_h\}$$

form a basis for R over $\mathbb{F}_o$. Since $(\mathcal{J}(R))^{k-1} = p^{k-1} R_o$ and

$$ann(\mathcal{J}(R)) = p^{k-1} R_o \oplus R_o u_1 \oplus ... \oplus R_o u_h,$$

then $(\mathcal{J}(R))^{k-1} \subset ann(\mathcal{J}(R))$. Moreover, there exists an element b in R of multiplicative order $p^r - 1$ such that $R_o = \mathbb{Z}_{p^k}[b]$ and $\mathbb{F}_o =< b > \cup(0)$(see e.g [1]). The following results on $(\mathcal{J}(R))^{k-1}$ and $ann(\mathcal{J}(R))$ are worth noting.

Lemma 3. *Let R be an R_o- module idealization with $u_i|_U$ and $\mathcal{J}(R)$ is the Jacobson radical of R. Then $1 + (\mathcal{J}(R))^{k-1} \cong (\mathcal{J}(R))^{k-1}$*

Proof. For the case $\sigma_i = id_{R_o}, i = 1, ..., h$, consider the map

$$\phi : 1 + (\mathcal{J}(R))^{k-1} \to (\mathcal{J}(R))^{k-1}$$

defined by

$$1 + \sum_{t=1}^{m} x_{1t} x_{2t} ... x_{(k-1)t} \to \sum_{t=1}^{m} x_{1t} x_{2t} ... x_{(k-1)t}$$

$$(x_{it} \in \mathcal{J}(R), 1 \leq i \leq k-1, 1 \leq t \leq m).$$

Clearly, ϕ is an isomorphism. The general case for σ_i follows from this. □

The following result demonstrates how a vector space can be obtained from the Jacobson radical $J(R)$ (see [3]).

Lemma 4. *Let R be a ring with property P and $\mathbb{F} = R/\mathcal{J}(R)$. Then $\mathcal{J}(R)/ann(\mathcal{J}(R))$ is a vector space over $\mathbb{F}$. Moreover, if the dimension $dim_{\mathbb{F}}(\mathcal{J}(R)/ann(\mathcal{J}(R))) = d$, then $dim_F(\mathcal{J}(R))^{k-1} \leq d^{k-1}$.*

Now, let R be an R_o- module idealization with $u_i|_U$ and with invariants p, n, r, k, h. Let $\{u_1, ..., u_h\}$ be the set of generators for U and $\{\sigma_1, ..., \sigma_h\}$ be the associated automorphisms of R with respect to the maximal Galois subring of R_o of R where σ_j occurs with multiplicity $m_j, j = 1, ..., h$

Now, suppose S is another ring of the same type with the same invariants constructed from the same maximal Galois subring R_o and $\{\widehat{\sigma_1}, ..., \widehat{\sigma_h}\}$ are the associated automorphisms of S with respect to R_o, where $\widehat{\sigma_i}$ occurs with multiplicity $\widehat{m_j}, j = 1, ..., h$. We determine the conditions under which the two rings, R and S are isomorphic.

Proposition 6. *With the above notation, $R \cong S$ iff there exists $\theta \in Aut(R_o)$, possibly distinct from $\{\sigma_1, ..., \sigma_h\} = \{\widehat{\sigma_1}, ..., \widehat{\sigma_h}\}$ and (after possible re-indexing), $m_j = \widehat{m_j}, j = 1, ..., h$*

Since R is generated by b and u_i, $1 \leq i \leq h$, it suffices to find the images of these elements for complete determination of the automorphisms. Also, every automorphism from R to R reduced to R_o fixes R_o. Evidently, to determine the automorphism group of R (Aut$R$), first we show that R_o is invariant under any automorphism $\phi \in$ Aut$R$. Next, we show that the ideal $\mathcal{J}^{k-1} = p^{k-1}R_o$ is also invariant under any automorphism $\phi \in$ Aut$R$. Finally, we compute the images of the rest of the generators, by a fixed element of Aut$R$ (see [9]). According to the authors in [10], finding an isomorphism θ from a ring R onto a ring S involves considering $T = R \oplus S$ and computing the generator set ζ of Aut(T). The following Lemma is due to Chikunji in [9]

Lemma 5. *Let $\theta \in Aut(R)$. Then $\theta(R_o)$ is a [11] maximal Galois subring of R and $\theta(R_o) = R_o$.*

Proof. Suppose $\theta : R \to R$ is an automorphism. Obviously, $\theta(R_o)$ is a maximal Galois subring of R so that there exists an element $z \in R$ such that $z\theta(R_o)z^{-1} = R_o$.

Now, consider the map $\varphi : R \to R$ given by $r \mapsto z\theta(r)z^{-1}$. Then φ is an automorphism of R which sends R_o to itself. □

According to the author in [3], the problem of finding automorphisms of a ring reduces to that of finding non trivial ring automorphisms. Now, we discuss the automorphisms of the R_o- module idealizations with $u_i|_U$. Let R be an R_o - module idealization with $u_i|_U$. Then $\mathcal{J}(R) = pR_o \oplus R_o u_1 \oplus ... \oplus R_o u_h$, $(\mathcal{J}(R))^{k-1} = p^{k-1}R_o$, annihilator $ann(\mathcal{J}(R)) = p^{k-1}R_o \oplus R_o u_1 \oplus ... \oplus R_o u_h$ and $(\mathcal{J}(R))^k = (0)$. We separately consider the cases where R is commutative and non commutative.

Case 1: R is noncommutative

For a noncommutative ring R, then there exists $z \in R^{\times}$ which does not commute with the whole of R, thus defines a non trivial automorphism. In this case, let p be a prime integer, then the map $\phi : R \to R$ defined by

$$\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j \to z((\sum_{i=0}^{k-1} a_i p^i)^{\sigma} + (\alpha_j^{\sigma} u_j))z^{-1}, \sigma \in Aut(R_o), a_i, \alpha_j \in \mathbb{F}_o$$

is a nontrivial automorphism of R.

Case 2: R is commutative

A commutative R is expressible in terms of its basis elements $\{b, u_1, ..., u_h\}$ so that $p^{k-1}u_i = 0, u_i u_j = p^2 \gamma_{ij}$ where $\gamma_{ij} \in \mathbb{F}_o, 1 \leq i, j \leq h$ and p is a prime integer. Let $m_i \in \mathbb{F}_o$. Each of the elements $p^{k-1}m_i \in (\mathcal{J}(R))^{k-1}, (1 \leq i \leq h)$ defines a nontrivial automorphism as follows:

$$\phi_1 = \begin{cases} u_1 \to u_1 + p^{k-1}m_1 \\ u_2 \to u_2 \\ \vdots \\ u_h \to u_h \end{cases}$$

$$\phi_2 = \begin{cases} u_1 \to u_1 \\ u_2 \to u_2 + p^{k-1}m_2 \\ \vdots \\ u_h \to u_h \end{cases}$$

$$\vdots$$

$$\phi_h = \begin{cases} u_1 \to u_1 \\ u_2 \to u_2 \\ \vdots \\ u_h \to u_h + p^{k-1}m_h \end{cases}$$

and the product $\phi = \phi_1\phi_2...\phi_h$ is an automorphism of R restricted to U.

The Group X

Let R be an R_o- module idealization with $u_i|_U$ and ξ be the number of nontrivial associated automorphisms of R with respect to R_o taken with their multiplicities η_i,

$$U_{\xi} = \sum_{\sigma_j \neq id_{R_o}} \oplus R_o u_j$$

and

$$U_{\eta} = \sum_{\sigma_j = id_{R_o}} \oplus R_o u_j, (1 \leq i, j \leq h).$$

Suppose $z \in 1 + \mathcal{J}(R)$ where $z = 1 + c + d$ with $c \in pR_o \oplus U_\eta = \mathcal{J}(R) \cap Z(R)$, $d \in U_\xi$. Let $X = \{\pi_z : z \in 1 + \mathcal{J}(R)\}$ and

$$\pi_z(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = z(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j)z^{-1}.$$

Define a map $f : X \to U_\xi$ by $f(\pi_z) = d$

Now, let $\pi_z = \pi_{s_1}$. Since $z \in 1 + \mathcal{J}(R)$ and $(\mathcal{J}(R))^k = (0)$, we write $z = s_1 . \prod_{k=2}^{n} s_k$ where

$$\prod_{k=2}^{n} s_k = s_1^{-1} z, \prod_{k=2}^{n} s_k \in 1 + \mathcal{J}(R).$$

If $s_1 = 1 + c + d$ and

$$\prod_{k=2}^{n} s_k = \prod_{\nu=1}^{n-1} 1 + t_\nu$$

where $d \in U_\xi$ and

$$t_\nu, ct_\nu, dt_\nu \in pR_o \oplus U_\eta = \mathcal{J}(R) \cap Z(R), 1 \leq \nu \leq n-1$$

then

$$z = \prod_{\nu=1}^{n} s_\nu = 1 + (c + \widehat{t} + c\widehat{t} + d\widehat{\widehat{t}}) + d$$

where $\widehat{t} = \sum_{\nu=1}^{n-1} t_\nu +$ sums of mixed products of $t_\nu, 1 \leq \nu \leq n-1$ and $\widehat{\widehat{t}} = \widehat{t} - \sum_{\nu=1}^{n-1} t_\nu$. Thus $g(\pi_z) = g(\pi_{s_1})$ showing that g is well defined. Since R is finite, g is clearly onto. Moreover, it is routine to show that $g(\pi_{z_1}\pi_{z_2}) = g(\pi_{z_1}) + g(\pi_{z_2})$. Thus

$$g : X \to U_\xi$$

is an isomorphism.

Let $U = U_1 \oplus ... \oplus U_s$, where each $U_\nu = \sum_{\sigma_k = \sigma_j} \oplus R_o u_k$, so that σ_k is the automorphism associated with u_k and $1 \leq \nu \leq s$. The following is essentially Remark 2 in [12].
Suppose $U_\xi = \sum_{\sigma_j \neq id_{R_o}} \oplus R_o u_j, U_\eta = \sum_{\sigma_j = id_{R_o}} \oplus R_o u_j$, then

$$U_\eta = \begin{cases} U_k \text{ for some } k \in \{1, ..., s\}, \text{if one of the associated automorphisms} \\ \quad \text{of } R \text{ with respect to } R_o \text{ is trivial} \\ (0) \text{ if none of the associated automorphisms of } R \text{ with respect to } R_o \text{ is trivial} \end{cases}$$

Given that $U_\eta = U_k$ for some $k = \{1, ..., s\}$, take $U_\eta = U_s$. Then $U_\xi = U_1 \oplus ... \oplus U_m$ where $m = s - 1$. So, we consider $U = \sum_{j=1}^{h} \oplus R_o u_j, h = l_1 + ... + l_s$ where each l_ν is the number of elements in the generating sets for U_ν, $(\nu = 1, ..., s)$

The following Lemmas are useful in determining the automorphisms of idealizations constructed in this paper and their proofs can be obtained in [11].

Lemma 6. *Let R be an R_o- module idealization with $u_i|_U$ and $G = Aut(R)$. Suppose $K \leq G$ containing all the automorphisms ϕ defined by*

$$\phi(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = (\sum_{i=0}^{k-1} a_i p^i)^\sigma + \sum_{\nu=1}^{h}\sum_{j=1}^{h} \alpha_j^\sigma \phi_\nu(u_j), a_i, \alpha_j \in \mathbb{F}_o$$

$\sigma \in Aut(R_o)$ and if $u_j \in U_\eta$, then $\phi_\nu \in Aut_{R_o/pR_o} U_\xi$.
Suppose K_1 is a subgroup of K containing automorphisms ϕ_σ defined by

$$\phi_\sigma(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = (\sum_{i=0}^{k-1} a_i p^i)^\sigma + \sum_{j=1}^{h} \alpha_j^\sigma u_j, a_i, \alpha_j \in \mathbb{F}_o$$

and K_2 be a subgroup of K containing automorphisms ϕ^* defined by

$$\phi^*(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = (\sum_{i=0}^{k-1} a_i p^i) + \sum_{\nu=1}^{s}\sum_{j=1}^{h} \alpha_j \phi_\nu u_j, a_i, \phi_\nu \in Aut_{R_o/pR_o} U_\nu$$

if $u_j \in U_\nu$. Then $K = K_2 \times_\tau K_1$ where $\tau : K_1 \to Aut(K_2)$ defined by $\phi_\sigma \to (\phi^* \to \phi_\sigma^*)$ is a group homomorphism.

Lemma 7. *Let R be an R_o- module idealization with $u_i|_U$ and $G = Aut(R)$. Suppose $L \leq G$ containing all the automorphisms π defined by*

$$\pi(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = z(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j + \sum_{\sigma_j = id_{R_o}} p^{k-1}\alpha_j m_j) z^{-1}, z \in R^*$$

$$(z = s_1 ... s_n, s_i \in 1 + \mathcal{J}(R), 1 \leq i \leq n), m_i, \alpha_j \in \mathbb{F}_o$$

σ_j is the automorphism associated with u_j.
Suppose L_1 is a subgroup of L containing automorphisms $\overline{\pi}$ defined by

$$\overline{\pi}(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = (\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j + \sum_{\sigma_j = id_{R_o}} p^{k-1}\alpha_j m_j, a_i, m_j \in \mathbb{F}_o$$

σ_j is the automorphism associated with u_j, and

L_2 be a subgroup of L containing automorphisms π_z defined by

$$\pi_z(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) = z(\sum_{i=0}^{k-1} a_i p^i + \sum_{j=1}^{h} \alpha_j u_j) z^{-1}, z \in R^\times.$$

Then $L = L_1 \times L_2$

Lemma 8. *Let K and L be as defined in Lemmas 6 and 7 respectively. Then $G = L \times_\alpha K$ where $\alpha : K \to Aut(L)$ defined by*

$$\phi^*\phi_\sigma \to (\overline{\pi}\pi_z \to \overline{\pi}_\sigma(\phi^*\phi_\sigma) z (\phi^*\phi_\sigma)^{-1})$$

is a group homomorphism.

Remark: Notice that each element of G is of the form $\pi\phi$, where $\pi \in L, \phi \in K$.

Lemma 9. *Let $f : L_1 \to U_\eta$ defined by $f(\overline{\pi}) = \sum_{\sigma_j = id_{R_o}} m_j u_j$. Then $L_1 \cong U_\eta$.*

Remark: Using the above Lemma and the fact that $L_2 = X = U_\xi$, we obtain $L_1 \times L_2 \cong U_\eta \oplus U_\xi$. Thus $U = U_1 \oplus ... \oplus U_s$, we can view each of the $U_\nu (\nu = 1, ..., s)$ as a vector space over R_o/pR_o and $Aut(U_\nu)$ corresponds to the set of invertible matrices $GL(l_\nu, R_o/pR_o)$. Now, $K_2 = Aut(U)$ corresponds to the set of the invertible matrices $\prod_{\nu=1}^{s} GL(l_\nu, R_o/pR_o)$. Since $Aut(R_o) \cong Aut(R_o/pR_o)$, notice that $K_1 = Aut(R_o/pR_o)$.
The following is the main result in this section. It is similar to Theorem 2 in [12].

Theorem 5. *Let R be an R_o- module idealization with $u_i|_U$. Then the map*

$$Aut(R) \to U \times_\alpha (\prod_{\nu=1}^{s} GL(l_\nu, R_o/pR_o) \times_\kappa Aut(R_o/pR_o))$$

is an isomorphism and in particular

$$\mid Aut(R) \mid = (p^r)^h (\prod_{\nu=1}^{s} \prod_{i=0}^{l_\nu - 1})((p^r)^{l_\nu} - (p^r))^i))r.$$

Proof. Use the fact that $\mid U \mid = (p^r)^h$, $\mid GL(l_\nu, R_o/pR_o) \mid = \prod_{i=0}^{l_\nu - 1}((p^r)^{l_\nu} - (p^r)^i)$, and $r = \mid Aut(R_o) \mid$, the order of the automorphism group of the ring R clearly follows. The rest of the proof follows from Lemmas 6 to 8. □

5 Zero Divisor Graphs of Idealization Rings

The zero divisor graphs discussed in this section were introduced by Anderson and Livingston [13]. They are denoted by $\Gamma(R)$. There are differences in the structural and geometrical properties of the graph $\Gamma(R)$ for each characteristic of the idealization ring. To this end, we present results on the graph $\Gamma(R)$ for each characteristic. Similar studies have been done in [14]. On some of the latest developments on the zero divisor graphs of finite commutative rings, the reader may refer to [15].

Case I: Zero Divisor Graph when char $R = p$.

Proposition 7. *Let R be an R_o- module idealization with $u_i|_U$. Suppose $charR = p$, then*

(i) $\Gamma(R) = K_{p^{hr}-1}$

(ii) $diam(\Gamma(R)) = 1$

(iii)

$$gr(\Gamma(R)) = \begin{cases} \infty \ if \ r = 1, h = 0 \ and \ p = 2, 3 \\ 3 \ elsewhere \end{cases}$$

(iv) $b(\Gamma(R)) = \infty$

Proof. (i) and (ii) easily follows from the fact that $\Gamma(R)$ is completely connected and has $p^{rh} - 1$ vertices. To establish (iii), if $r = 1, h = 0, 1$ and $p = 2, 3$ then $p^{hr} - 1 \leq 2$ so that $gr(\Gamma(R)) = \infty$. Otherwise, $p^{hr} - 1 > 2$ and since the completeness of the graph implies that $gr(\Gamma(R)) = 2diam(\Gamma(R)) + 1$, the result easily follows. To establish (iv), we use the formula $b(\Gamma(R)) = \frac{|N(S)|}{|S|}$. Since the set S of minimal degree in $V(\Gamma(R))$ is empty, the result easily follows. □

Case II: Zero Divisor Graph when char $R = p^2$.

Proposition 8. *Let R be an R_o- module idealization with $u_i|_U$. Suppose $charR = p^2$, then*

(i) $\Gamma(R) = K_{p^{(h+1)r}-1}$

(ii) $diam(\Gamma(R)) = 1$

(iii)

$$gr(\Gamma(R)) = \begin{cases} \infty \ if \ r = 1, h = 0 \ and \ p = 2, 3 \\ 3 \ elsewhere \end{cases}$$

(iv) $b(\Gamma(R)) = \infty$

Proof. Similar to Proposition 7. □

Proposition 9. *Let R be an R_o- module idealization with $u_i|_U$. Suppose $charR = p$ or p^2. Then $\Gamma(R)$ is triangular if R is one of the following.*

(i) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(ii) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$

(iii) $\mathbb{Z}_4[x]/ < x^2 + x + 1 >$

Proof. Fix the invariants p, r, h, k for which $\Gamma(R) = K_3$. □

Case III: Zero Divisor Graph when $charR = p^k, k \geq 3$.

Proposition 10. *Let R be an R_o- module idealization with $u_i|_U$. Suppose $char(R) = p^k, k \geq 3$ then*

$$gr(\Gamma(R)) = \begin{cases} p^{(\frac{k}{2}+h)r} - partite \ if \ k \ is \ even \\ p^{(\frac{k-1+2h}{2})r} - partite \ if \ k \ is \ odd \end{cases}$$

Proof. Refer to [14] □

Proposition 11. *Let R be an R_o- module idealization with $u_i|_U$. Suppose $charR = p^k, k \geq 3$ then*

(i) $diam(\Gamma(R)) = 2$

(ii) $gr(\Gamma(R)) = 3$

(iii)

$$b(\Gamma(R)) = \begin{cases} \frac{p^{(\frac{k}{2}+h)r}-2}{p^{(k-1+h)r}-p^{(\frac{k}{2}+h)r}} \ if\ k\ is\ even \\ \frac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}} \ if\ k\ is\ odd \end{cases}$$

Proof. Refer to [14] □

Proposition 12. *There exists no ring in the Construction whose zero divisor graph, $\Gamma(R)$ is an $n-$ gon, $n > 3$.*

Proof. Suppose R is not a field and the cardinality of $V(\Gamma(R))$ is greater than 3, then $\text{Ann}(Z(R)) \neq (0)$, since $Z(R)$ is a nilpotent ideal. If $0 \neq a \in Ann(Z(R))$, then a is adjacent to every other vertex in $V(\Gamma(R))$. □

Proposition 13. *Let R be a commutative ring with property P. Then $diam(\Gamma(R)) = 0, 1$ or 2.*

Proposition 14. *Let R be a commutative ring with property P. Then $gr(\Gamma(R)) = \infty$ if R is one of the following.*

(i) $R = \mathbb{Z}_2[x]/ < f(x) >$ *where f(x) is a monic irreducible polynomial modulo p of degree r*

(ii) $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(iii) $R = \mathbb{Z}_3 \oplus \mathbb{Z}_3$

6 Conclusion

The Galois ring module idealization has an interesting structure. For the finite commutative case, the unit group is expressible as a direct product of cyclic groups. The automorphism group of the idealization ring is a semi direct product of the Galois ring module by a general Linear Group over the field R_o/pR_o by the automorphism group of R_o/pR_o. The zero divisors of the idealization ring constructed in this paper form a unique maximal ideal thereby implying that the ring is completely primary. The results obtained in this study contribute significantly to the existing literature on the classification of finite rings. This is evident, because the rings considered in this work are completely primary and every commutative finite ring is expressible as a direct sum of completely primary finite rings. Furthermore, the results can be used to settle some questions in

matrix theory and the theory of algebraic cryptography. The future direction of this research should delve into the matrices of the zero divisor graphs of the rings with property P.

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Competing Interests

Author has declared that no competing interests exist.

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