On the Zero Divisor Graphs of Finite Rings in Which the Product of Any Two Zero Divisors Lies in the Coefficient Subring

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Abstract

Let r be a positive integer and $2 \le k \in \mathbb{Z}$. Let $GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k. Consider, $R = GR(p^{kr}, p^k) \oplus U$ where U is a finitely generated $GR(p^{kr}, p^k)$ module. If $Z(R)$ is the set of zero divisors in R satisfying the condition $(Z(R))^2 \subseteq GR(p^{kr}, p^r)$ then it is well known that R is a completely primary finite ring and the structure of its group of units has been studied before. In this paper, we study the structure of its zero divisors via the zero divisor graphs.

Keywords: Finite rings, Zero divisor graphs.

1. Introduction

The set of the nonzero zero divisors in R is denoted by $Z(R)^*$. Also $\Gamma(R)$ denotes the graph of the nonzero zero divisors in which two vertices u, v are adjacent iff $uv = 0$. We shall also explore some cases of the graph $\Gamma_E(R)$ whose vertices are equivalent classes of zero divisors. Two zero divisors, u

and v in R are said to be equivalent if the annihilators, $Ann_R(u) = Ann_R(v)$ and two distinct classes

u and v in $\Gamma_E(R)$ are adjacent iff $uv = 0$ in R. For more information on $\Gamma(R)$ and $\Gamma_E(R)$, the

reader may refer to [1, 2, 3, 5]. It is believed that these graphs provide a better description of the structures

of the zero divisors.

2. Rings in which $(Z(R))^2 \subseteq GR(p^{kr}, p^k)$

2.1. The Construction

Let r be a positive integer and $2 \leq k \in \mathbb{Z}$. Let $R' = GR(p^{kr}, p^k)$ be a Galois ring. For each

$i = 1, \cdots, h$, let $u_i \in Z(R)$ such that U is an $h-$dimensional R' module so that $R = R' \oplus U$ is an

additive Abelian group. On R define multiplication as follows; For $r', s' \in R', \alpha_i, \omega_i, \lambda_{ij} \in R' / pR'$

and $\sigma_i \in Aut(R')$, let $\left(r' + \Sigma_{i=1}^h \alpha_i u_i\right)\left(s' + \Sigma_{i=1}^h \omega_i u_i\right) = r's' + p^{k-1}\Sigma_{i,j=1}^h \lambda_{ij}\left(\alpha_i(\omega_j)^{\sigma_i} + pR'\right) +$

$\Sigma_{i=1}^h \left[(r' + pR')\omega_i + \alpha_i(s' + pR')^{\sigma_i}\right] u_i$.

This multiplication turns R into a ring with identity $(1, 0, 0, \cdots, 0)$.

Lemma 2.1. Let R be a ring in which $(Z(R))^2 \subseteq GR(p^{kr}, p^k)$. Then, R is commutative if and

only if $\sigma_i = id_{R'}$, (the identity automorphism) for every $i = 1, \cdots, h$.

Proposition 2.2. Let R be a ring constructed above. Then, it is a completely primary finite ring of

characteristic p^k satisfying:

(i). $Z(R) = pR' \oplus U$

(ii). $\left(Z(R)\right)^{k-1} = p^{k-1}R'$

(iii). $\left(Z(R)\right)^k = (0)$

In the sequel, we provide results on the graphical properties of the zero divisors of the commutative

rings constructed in Section 2.1.

2.2. Graphs of Nonzero Zero Divisors

Denoted by $\Gamma(R)$, in this graph, the nonzero zero divisors are considered vertices and two vertices

u, v are adjacent iff $uv = 0$. These graphs were introduced by Anderson and Livingston in [2].

2.3. Results

Proposition 2.3. Let R be a commutative ring constructed above. Then, the graph of R, $\Gamma(R)$

satisfies the following:

(i). $\Gamma(R) = p^{(h+k-1)r} - 1$

(ii). $diam(\Gamma(R)) = 2$

(iii). $gr(\Gamma(R)) = \begin{cases} \infty, & r = 1, h = 1, p = 2. \\ 3, & elsewhere \end{cases}$

(iv). The binding number, $b(\Gamma(R)) = \begin{cases} \dfrac{p^r - 1}{p^{(1+h)r} - p^r} & \text{if } k = 2 \\[2ex] \dfrac{p^{(\frac{k}{2})r} - 1}{p^{(k-1+h)r} - p^{(\frac{k}{2})r}} & \text{if } k \geq 3 \text{ and even} \\[2ex] \dfrac{p^{(\frac{k-1}{2})r} - 1}{p^{(k-1+h)r} - p^{(\frac{k-1}{2})r}} & \text{if } k \geq 3 \text{ and odd} \end{cases}$

Proof.

(i). Clearly $Z(R) = pR' \oplus U$. Since $Z(R)$ is a maximal ideal of R, the quotient $R/Z(R)$ is

a field of order p^r. Now consider $0 \neq a \in R/Z(R)$, then $(R/Z(R))^* = <a>$ and

$o(a) = p^r - 1$. This shows that each element which does not belong to $Z(R)$ has an inverse.

Thus $Z(R) = p^{(h+k-1)r}$ and $Z(R)^* = p^{(h+k-1)r} - 1$.

(ii). The annihilator, $Ann(Z(R)) = p^{k-1}R'$. Now , let $x \in Z(R)^*$ but $x \notin p^{k-1}R'$, then there

exists $y \in Z(R)^*$ such that $xy \in pR'$. But $xyz = 0$, where $z \in Ann(Z(R)) = pR'$. So

$diam(\Gamma(R)) = 2$.

(iii). If $r=1, p=2, h=1$, then $Z(R)^\star = \{(0,1),(2,0),(2,1)\}$. In this case $\Gamma(R)$ is a claw graph, since $(2,0)$ is adjacent to the other two vertices while $(0,1)$ and $(2,1)$ are non adjacent.

Elsewhere, $\left(Ann(Z(R))\right)^\star = p^r \geq 2$.

Now, let $x,y \in \left(Ann(Z(R))\right)^\star$, then x and y are adjacent. Moreover, any $z \in Z(R)^\star$ is adjacent to x and y. This completes the proof.

(iv). We consider the three cases separately.

Case I: $k=2$. Consider $N(S) = Ann(Z(R))^\star = pR^{'} \setminus \{0\}$. So $N(S) = p^r - 1$. Now, $S = V(\Gamma(R)) \setminus N(S)$, so that $S = p^{(h+1)r} - 1 - (p^r - 1)$. Thus $b(\Gamma(R)) = \frac{N(S)}{S} = \frac{p^r - 1}{p^{(h+1)r} - p^r}$.

Case II: $k \geq 3$ *and even.* Let $X_i = \{\Sigma_{i=1}^r a_i \lambda_i\}, 1 \leq i \leq r, a_i \in \left\{0, j\left(p^{\frac{k}{2}}\right)\right\}, 1 \leq j \leq p^{\frac{k}{2}} - 1$, then define $V_{\Sigma_i a_i \lambda_i} = X_i \setminus \{0\}$ and $V_1 = Z(R)^\star \setminus \bigcup_i V_{\Sigma_i a_i \lambda_i}$. From the definition of V_1, $N(V_1) = \bigcup_i V_{\Sigma_{a_i \lambda_i}}$. So $|N(V_1)| = p^{(\frac{k}{2})r} - 1$. Also, $|V_1| = |Z(R)^\star - N(V_1)| = |Z(R)^\star| - |\bigcup_i V_{\Sigma_{a_i \lambda_i}}| = p^{(k-1+h)r} - 1 - (p^{(\frac{k}{2})r} - 1) = p^{(k-1+h)r} - p^{(\frac{k}{2})r}$. Using the ratio, $\frac{|N(V_1)|}{|V_1|}$, we obtain the binding number.

Case III: $k \geq 3$ *and odd.* Let $X_i = \{\Sigma_i^r a_i \lambda_i\}, 1 \leq i \leq r, a_i \in \left\{0, (j-1)p^{\frac{k+1}{2}}\right\}, 2 \leq j \leq p^{\frac{k-1}{2}}$. Then define $V_{\Sigma_i a_i \lambda_i} = X_i \setminus \{0\}, V_1 = Z(R)^\star \setminus \bigcup_i V_{\Sigma_i a_i \lambda_i}$. From the definition of V_1, $N(V_1) = \bigcup_i V_{\Sigma_{a_i \lambda_i}}$, so that $|N(V_1)| = p^{(\frac{k-1}{2})r} - 1$. Also, $|V_1| = |Z(R)^\star - N(V_1)| = |Z(R)^\star| - |\bigcup_i V_{\Sigma_{a_i \lambda_i}}| = p^{(k-1+h)r} - 1 - (p^{(\frac{k-1}{2})r} - 1) = p^{(k-1+h)r} - p^{(\frac{k-1}{2})r}$. The result follows from the ratio $\frac{|N(V_1)|}{|V_1|}$.

Proposition 2.4. Let R be a ring constructed in Section 2. If $h=1$, then

$$\Gamma(R) = \begin{cases} p^{(\frac{k}{2})r} - partite & \text{if } k \text{ is even} \\ p^{(\frac{k-1}{2})r} - partite & \text{if } k \text{ is odd} \end{cases}$$

Proof. Consider $\lambda_1, \cdots, \lambda_r \in R^{'}$ with $\lambda_1 = 1$ such that $\lambda_1, \cdots, \lambda_r \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $\mathbb{F}_p$. Since the two cases do not overlap, we

treat them in turn.

Case I: k is an even integer.

Let $X_i = \{\Sigma_{i=1}^r a_i \lambda_i\}, 1 \leq i \leq r, a_i \in \left[0, j\left(p^{\frac{k}{2}}\right)\right], 1 \leq j \leq p^{\frac{k}{2}} - 1$, then $Z(R)^\star$ is partitioned into the following subsets;

$$V_{\Sigma_i a_i \lambda_i} = X_i \setminus \{0\}, V_1 = Z(R)^\star \setminus \bigcup_i V_{\Sigma_i a_i \lambda_i} \text{ and } Z(R)^\star = V_1 \bigcup \left(\bigcup_i V_{\Sigma_i a_i \lambda_i}\right).$$

The subsets are nonempty, mutually disjoint and contain nonadjacent vertices. Moreover, $\bigcup_i V_{\Sigma_i a_i \lambda_i} = p^{\left(\frac{k}{2}\right)r} - 1$ so that $\Gamma(R)$ is $p^{\left(\frac{k}{2}\right)r}$ partite.

Case II: k is an odd integer.

Let $X_i = \{\Sigma_i^r a_i \lambda_i\}, 1 \leq i \leq r, a_i \in \left[0, (j-1) p^{\frac{k+1}{2}}\right], 2 \leq j \leq p^{\frac{k-1}{2}}, Z(R)^\star$ is partitioned into the following mutually disjoint subsets;

$$V_{\Sigma_i a_i \lambda_i} = X_i \setminus \{0\}, V_1 = Z(R)^\star \setminus \bigcup_i V_{\Sigma_i a_i \lambda_i}$$

The subsets are nonempty and each contains nonadjacent vertices. In addition, $\bigcup_i V_{\Sigma_i a_i \lambda_i} = p^{\left(\frac{k-1}{2}\right)r} - 1$ so that $\Gamma(R)$ is $p^{\left(\frac{k-1}{2}\right)r}$ partite.

Consequently, we obtain the following.

Proposition 2.5. Let R be a ring considered in Proposition 2.4. Then the clique number of $\Gamma(R)$ is given by

$$\omega(\Gamma(R)) = \begin{cases} p^{\left(\frac{k}{2}\right)r} & k \text{ is even;} \\ p^{\left(\frac{k-1}{2}\right)r} & k \text{ is odd.} \end{cases}$$

Proof. The clique number coincides with the number of partite subsets since each subset of vertices has at least a vertex which is adjacent to another vertex in a distinct subset.

Proposition 2.6. If $r = 1$, $k \geq 3$, $\alpha \in (R')^*$ and $u_i \in U$ then,

(i) $deg(x) = \begin{cases} p^{k-1+h} - 2, & \text{if } x \in Ann(Z(R))^*; \\ p^{k+h-2} - 1, & \text{if } x \in Ann(Z(R)) \oplus <u_i>. \end{cases}$

(ii) $deg(p^l \alpha) = \begin{cases} p^{l+h} - 1, & \text{if } 2l < k; \\ p^{l+h} - 2, & \text{if } 2l \geq k. \end{cases}$

(iii) $deg(p^l\alpha+u_i) = \begin{cases} p^{l+h-1}-1, & \text{if } 2l < k; \\ p^{l+h-1}-2, & \text{if } 2l \geq k. \end{cases}$

Proof.

(i) Since $Z(R) = pGR(p^k,p^k)\oplus U$, it follows that each vertex in $Ann(Z(R))^*$ is adjacent to

every other vertex in $Z(R)$ except 0. Now, $Z(R)$ has p^{k-1+h} elements which implies that the number

of edges incident to x are $|Z(R)|-2$. Now, let

$$x \in Ann(Z(R))\oplus <u_i>.$$

Then, x is adjacent to every other vertex in $Z(R)$ except 0 and the case where $u_i = u_j$. Since

$$|Z(R)| = p^{k-1}\cdot p^h,$$

the exclusion of the vertices leaves p^{k-2+h} vertices incident to x.

(ii) The g.c.d $(\alpha,p^k) = 1$. So, the vertices adjacent to $p^l\alpha$ in the graph $\Gamma(R)$ are not distinct

from the vertices adjacent o p^l. So, it suffices to find the vertices to find the vertices adjacent to p^l.

Now, the coefficient vertices adjacent to p^l may be arranged in the sequence

$$p^{k-l}, 2p^{k-l}, ..., p^{k-l}+(n-1)p^{k-l}$$

where n is the number of adjacent vertices. Since these vertices are adjoined to U and

$$p^{k-l}+(n-1)p^{k-l} = p^k - p^{k-l},$$

then $n = p^{l+h}-1$ if $p^{k-l} > p^l$ or $k > 2l$

Next, if $p^l \geq p^{k-l}$ or $2l \geq k$, then

$$p^l = p^{2l-k}(p^{k-l})$$

which is adjacent to itself. Considering the fact that the coefficient vertices are adjoined to U, then

$$deg(p^l) = p^{l+h}-2.$$

(iii) The argument is similar to (ii), except that in this case $u_i \neq 0$, so that the only vertices in $\Gamma(R)$ which are adjacent to $p^l \alpha + u_i$ are obtained when $u_i \neq u_j$. Therefore, only p^{h-1} vertices of U are considered. Since the coefficient vertices adjacent to $p^l \alpha + u_i$ are $p^l - 2$; the result easily follows.

In the section, we investigate the graphs determined by the equivalence classes.

2.4. Zero Divisor Graphs Determined by the Equivalence Classes $(\Gamma_E(R))$

These graphs were introduced by S.B Mulay [4]. Two nonzero zero divisors x and y in a ring R are said to be equivalent if the annihilators, $Ann_R(x) = Ann_R(y)$. The equivalence class of x is denoted by $[x]$. The graph determined by the equivalence classes is denoted by $\Gamma_E(R)$ and $V(\Gamma_E(R)) = \{[x] \mid x \in Z(R)^*\}$. Two distinct classes x and y in $\Gamma_E(R)$ are adjacent iff $xy = 0$ in R. In this section, we present some results on the zero divisor graphs determined by the equivalence classes in the ring R given by the construction.

2.5. Results

Proposition 2.7. Let R be a commutative ring given by the construction. Then

$$\Gamma_E(R) = \begin{cases} \dfrac{k}{2} & \text{partite if } k \text{ is even;} \\ \dfrac{k+1}{2} & \text{partite if } k \text{ is odd.} \end{cases}$$

Proof.

Case I: k is even: We partition $\Gamma_E(R)$ into the following subsets

$$V_1 = \{(Z(R))^i \mid 1 \leq i \leq \frac{k}{2}\},$$

$$V_j = \{(Z(R))^j\}, \frac{k}{2} < j < k-1$$

For each j, $V_1 \cap V_j = \varnothing$ and V_j are mutually disjoint. Moreover,

$$V_1 \bigcup \bigcup_j \{V_j\} = \Gamma_E(R)\}.$$

Upon counting the disjoint subsets, we obtain the results.

Case II: k is odd: $\Gamma_E(R)$ is partitioned into the following subsets

$$V_1 = \{(Z(R))^i \mid 1 \le i \le \frac{k-1}{2}\}, V_j = \{(Z(R))^j\}, \frac{k-1}{2} < j < k-1$$

For each j, $V_1 \cap V_j = \varnothing$ and V_j are mutually disjoint. Moreover,

$$V_1 \bigcup \bigcup_j \{V_j\} = \Gamma_E(R)\}.$$

Upon counting the disjoint subsets, we obtain the result.

As a consequence of the Proposition, we obtain the following result.

Corollary 1. Let $\Gamma_E(R)$, be the graph determined by the equivalence classes of the zero divisors of

a commutative ring given by the construction in Section 2.1. Then, the clique number of the graph is given

by

$$\omega(\Gamma_E(R)) = \begin{cases} \dfrac{k}{2} & \text{if } k \text{ is even;} \\ \dfrac{k+1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 2.8. Let R be a ring given by the construction. Then

(i) $diam(\Gamma_E(R)) \begin{cases} 0, & k = 2; \\ 1, & k = 3; \\ 2, & \text{elsewhere.} \end{cases}$

(ii) $gr(\Gamma_E(R)) = \begin{cases} \infty, & k = 2,3; \\ 3, & k > 3. \end{cases}$

(iii) $gr(\Gamma_E(R)) = \begin{cases} 0, & k = 2; \\ 1, & k = 3; \\ \frac{k-4}{k}, & k \ge 4 \text{ and even;} \\ \frac{k-5}{k-1}, & k \ge 4 \text{ and odd.} \end{cases}$

Proof.

(i) If $k = 2$, then $\Gamma_E(R)$ has a single vertex. If $k = 3$, then $\Gamma_E(R)$ is a straight edge

$[Z(R)] - [(Z(R))^2]$. Now, for $k > 3$, $[(Z(R))^{k-1}]$ is adjacent to every other vertex in $\Gamma_E(R)$.

But $\Gamma_E(R)$ is not complete because, for $0 < i < \frac{k}{2}$ where k is even and $0 < i < \frac{k-1}{2}$, when k

is odd, there exists $j > i > 0$ so that $[(Z(R))^i][(Z(R))^{k-1-j}] = [(Z(R))^{k-1+i-j}] \neq 0$.

(ii)   For $k = 2$ or $k = 3$,   the   result   follows   from   (i).   Elsewhere,

$[(Z(R))^{k-1}] - [(Z(R))^i] - [(Z(R))^{k-i}] - [(Z(R))^{k-1}]$ is a cycle, for $0 < i < \frac{k}{2}$ where k is even

and $0 < i < \frac{k-1}{2}$ when k is odd. It is important to note that there exists no $n - \text{gon}$, $n > 3$

because $[(Z(R))^{k-1}]$ is adjacent to every other vertex in $\Gamma_E(R)$

(iii) If $k = 2$, then $S = Z(R)$ and $N(S) = \varnothing$. So, $|S| = 1$ and $|N(S)| = 0$. If $k = 3$, then

$|N(S)| = 1$. Now, let $k > 3$. When k is even $|V_1| = \frac{k}{2}$ while $|N(V_1)| = \frac{k-4}{2}$. When k is odd,

then $|V_1| \frac{k-1}{2}$ while $|N(V_1)| = \frac{k-5}{2}$. Using the ratio $\frac{|N(V_1)|}{|V_1|}$ for the binding number, we obtain the

result.

Proposition 2.9. Let $x \in \Gamma_E(R)$, then $deg(x) = 1$ when $k = 2$ or 3 and for $k \geq 4$

$$deg(x) = \begin{cases} i - 1, x = [(Z(R))^i], i \geq \dfrac{k}{2} & k \text{ is even}; \\[2mm] i, x = [(Z(R))^i], i < \dfrac{k}{2} & k \text{ is even}; \\[2mm] i - 1, x = [(Z(R))^i], i \geq \dfrac{k+1}{2} & k \text{ is odd}; \\[2mm] i, x = [(Z(R))^i], i < \dfrac{k+1}{2} & k \text{ is odd}. \end{cases}$$

Proof. Let $k = 2$ or 3, then $\Gamma_E(R)$ is a straight edge $[Z(R)] - [(Z(R))^2]$. Now, suppose

$k \geq 4$. If k is even and $i < \frac{k}{2}$, then, every $[(Z(R))^i]$ in $\Gamma_E(R)$ is adjacent to some $[(Z(R))^{k-j}]$,

$1 \leq j \leq i$, so the degree $deg([(Z(R))^i]) = i$. If k is odd and $i < \frac{k+1}{2}$, we obtain a similar result.

Now, let k be even and $i \geq \frac{k}{2}$, then each $[(Z(R))^i]$ is adjacent to some $[(Z(R))^i]$,

$k - i \leq j \leq k - 1$. If $i \neq j$, then $deg([(Z(R))^i]) = i - 1$. The case when k is odd and $i \geq \frac{k+1}{2}$ is

proved in a similar manner.

3. Conclusion

In this paper, we investigated the structures of the zero divisor graphs of the rings described by the Construction in Section 2. There are quite a great deal of gaps to be filled in future research. It is noteworthy, that some results have been provided only for specific cases. Further research should dwell on generalizations of the graphical properties of the zero divisors of the constructed rings.

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