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Article *in* Journal of Advances in Mathematics and Computer Science · December 2020

DOI: 10.9734/jamcs/2020/v35i830316

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Journal of Advances in Mathematics and Computer Science

35(8): 83-90, 2020; Article no.JAMCS.63771

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Automorphisms of Zero Divisor Graphs of Cube Radical Zero Completely Primary Finite Rings

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Authors' contributions

This work was carried out in collaboration among all the authors. LHM managed to write down the manuscripts while OMO and OMO managed supervision of the manuscripts. All the authors have read through the manuscripts.

Article Information

DOI: 10.9734/JAMCS/2020/v35i830316

Editor(s):

(1) Dr. Sheng Zhang, Bohai University, China.

Reviewers:

(1) Caio Cesar Cardoso da Silva, Federal University of Rio Grande do Sul, Brazil.

(2) Kenmogne Fabien, University of Douala, Cameroon.

(3) Ram Naresh Saraswat, Manipal University Jaipur, India.

Complete Peer review History: http://www.sdiarticle4.com/review-history/63771

Received: 10 October 2020

Accepted: 14 December 2020

Published: 24 December 2020

Original Research Article

Abstract

One of the most interesting areas of research that has attracted the attention of many scholars are theory of zero divisor graphs. Most recent research have focused on properties of zero divisor graphs with little attention given on the automorphsisms, despite the fact that automorphisms are useful in interpreting the symmetries of algebraic structure. Let R be a commutative unital finite rings and $Z(R)$ be its set of zero divisors. In this study, the automorphisms zero divisor graphs of such rings in which the product of any three zero divisor is zero has been determined.

Keywords: Automorphisms; zero divisor graphs; completely primary finite rings.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

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1 Introduction

The classification of automorphisms of graphs would have been exhausted if it was possible to find necessary and sufficient conditions to determine the full automorphism group. The classification is still open even though it has been done for some families of graphs. Graphs and graph automorphisms are two important structures studied in mathematics. Interestingly the theory of graphs and graph automorphisms are deeply connected. For instance, Evariste Galois characterized the general quintic univariate polynomial f over rationals by showing that the root of such polynomial cannot be expressed interms of radicals via automorphisms of structures of the splitting fields of f. A little account on automorphisms can be mentioned. Ojiema et'al [1] did considerable work on automorphisms unit groups of square radical completely primary finite rings. The research on automorphisms of zero divisor graphs of Galois rings was extensively done by Lao eta'l [2] while Ojiema et'al [3] characterized automorphisms of unit groups of power four radical zero finite commutative completely primary rings. Other modes of structural classifications of automorphisms can also be mentioned. The research on automorphisms of zero divisor graphs of square radical zero commutative finite ring was carried out by [4] where detail studies of structures and order formulae for automorphisms was invigorated setting stage for classification of automorphisms of rings of characteristics greater than 2.

One of the most interesting and extensively studies was to determine when two graphs are isomorphic. The completely primary finite rings having 3-nilpotent radical of Jacobson have been widely studied (see for example [[5], [6], [7], [8], [9]]) under various conditions based on some well chosen invariants. For instance in Chikunji [5], the study determined the automorphisms of such rings of characteristic p given $s,\ t,\ \lambda$ as invariants and h as the dimension of the submodules of the maximal Galois subring R_0. For the automorphisms of some classes of rings with characteristics $p^2,\ p^3$, reference can be made to [10]. The studies mentioned however, concentrated on the automorphisms of the the classes of the rings, leaving out the automorphisms of the graphs of the rings. Other modes of structural classification of the cube radical zero completely primary finite rings have been advanced in [9] among others. In fact, by construction, Chikunji in [9], considered the parameters t and s to be related by $t \leq \frac{s(s+1)}{2}$. It is well known that in such rings, an element is either a zero divisor or a unit. However, from the available literature on the determination and characterization of units groups of these rings, the following cases have been considered:

(i) $s = 2, t = 1, \lambda = 0, charR = p, p^2$ or p^3

(ii) $s = 2, t = 1, \lambda \geq 1, charR = p, p^2$ or p^3

(iii) $s = 2, t = 2, \lambda = 0, charR = p, p^2$ or p^3

(iv) $s = 3, t = 1, \lambda \geq 1, charR = p$

(v) $t \leq \frac{s(s+1)}{2}, \lambda = 0, charR = p, p^2$ or p^3

(vi) $t \leq \frac{s(s+1)}{2}, \lambda \geq 1, charR = p, p^2$ or p^3.

Let $R_0 = GR(p^{nr}, p^n)$ be Galois ring of order p^{nr} and characteristic p^n. Suppose U, V and W are R_0- modules generated by s, t and λ elements respectively. Consider the sets of commuting indeterminates $\{u_1, u_2, \cdots, u_s\}$, $\{v_1, v_2, \cdots, v_t\}$ and $\{w_1, w_2, \cdots, w_\lambda\}$ to be the generators of U, V and W respectively so that $R = R_0 \oplus U \oplus V \oplus W$ is an additive abelian group. Depending on the characteristic of R_0, we define suitable multiplication that turns R into a commutative ring wit identity and proceed to characterize the automorphisms of zero divisor graphs of a ring R, for the cases where $s = 2,\ t = 1,\ \lambda = 0$ since such rings have been constructed for all the characteristics.

2 Construction of Cube Radical Zero Finite Rings of Characteristic p

The following construction can be obtained from [9].

Let $R_0 = GF(p^r)$ be a Galois field. Suppose $\{u_1, u_2\}$ and $\{v\}$ are generating sets for R_0- modules U and V respectively, so that$R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ is an additive abelian group . On this group define multiplication as follows:

$(a_0, a_1, a_2, a_3)(b_0, b_1, b_2, b_3) = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, a_0b_3 + a_3b_0 + a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2)$. It is well known that this multiplication turns R into a commutative ring with identity $(1, 0, 0, 0)$.

Proposition 2.1. *Let R be a ring of the above Construction. The set of zero divisors $Z(R)$ satisfies the following:*

(i) $Z(R) = R_0u_1 \oplus R_0u_2 \oplus R_0v$.
(ii) $(Z(R))^2 = R_0v$.
(iii) $(Z(R))^3 = (0)$.

Proof. That the characteristic of R is p follows from the fact that char$R$=charR_0. We want to show that any element not in $R_0u_1 \oplus R_0u_2 \oplus R_0v$ is a unit. Let $a_0 \neq 0$, We determine the inverse of (a_0, a_1, a_2, a_3), say (b_0, b_1, b_2, b_3). From the multiplication in R, we need that $a_0b_0 = 1$, $a_0b_1 + a_1b_0 = 0$,and $a_0b_2 + a_2b_0 = 0$, $a_0b_3 + a_3b_0 + a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2 = 0$ which implies that $b_0 = {a_0}^{-1}$,$a_0b_1 = -a_1b_0 = -a_1{a_0}^{-1} \Rightarrow b_1 = -a_1{a_0}^{-2}$, $b_2 = -a_2{a_0}^{-2}$ and $b_3 = -a_3{a_0}^{-2} - {a_1}^2{a_0}^{-3} - a_1a_2{a_0}^{-3} - a_2a_1{a_0}^{-3} - {a_2}^2{a_0}^{-3}$. Therefore $(a_0, a_1, a_2, a_3)^{-1} = ({a_0}^{-1}, -a_1{a_0}^{-2}, -a_1{a_0}^{-2}, -a_3{a_0}^{-2} - {a_1}^2{a_0}^{-3} - a_1a_2{a_0}^{-3} - a_2a_1{a_0}^{-3} - {a_2}^2{a_0}^{-3}$.)

Properties (ii) and (iii) easily follows from the given multiplication. □

Some properties of the zero divisor graphs of R are given in the next proposition

Proposition 2.2. *Let R be a ring of the construction given in this section. Then*
(i) $|V(\Gamma(R))| = p^{3r} - 1$
(ii) $\Gamma(R)$ *is incomplete*
(iii) $diam(\Gamma(R)) = 2$

Proof. *(i)* Since $char = p, pu_1 = pu_2 = pv = 0$. So $|R_0u_1| = |R_0u_2| = |R_0v| = p^r$. Therefore $|Z(R)| = p^{3r}$ while $|Z(R)^*| = p^{3r} - 1 = |V(\Gamma(R))|$.
(ii) Follows from the fact that $(Z(R))^2 \neq 0$
(iii) Since $Ann(Z(R)) = (Z(R))^2$ and $\Gamma(R)$ is incomplete, there exist non adjacent $x, y \in V(\Gamma(R))$ so that for some $z \in Ann(Z(R))$, $x - z - y$ is the longest path in the graph □

The following result summarizes the structure of the automorphisms of the zero divisor graph of the ring constructed in this section.

Proposition 2.3. *Let $R = R_0u_1 \oplus R_0u_2 \oplus R_0v$ be a ring constructed in this section. Then* $Aut(\Gamma(R_0u_1 \oplus R_0u_2 \oplus R_0v)) \cong S_{p^{3r}-p^{2r}} \times S_{p^{2r}-1}$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0$ form a basis for R_0 over its prime subfield. It suffices to partition $Z(R)^*$ into mutually disjoint sets each of whose members have the same degree. Since $|Z(R)| = p^{3r}$ and $|Z(R)^*| = p^{3r} - 1$, each nonzero element of $Ann(Z(R))$ is of degree $p^{3r} - 2$. Let $a, b, c \in R_0$. Then, $Z(R) = \{a\xi_iu_1 + b\xi_iu_2 + c\xi_iv \mid a, b, c \in R_0\}$ and, $Ann(Z(R)) = \{a\xi_iu_1 + b\xi_iu_2 + c\xi_iv \mid a + b \equiv 0(\text{mod } p)\}$.

Now, in $ann(Z(R)), |\{(a,b) : a,b \in R_0\}| = p^r$ while $ann(Z(R)), |\{(a,b,c) : a,b,c \in R_0\}| = p^{2r}$. So, $|Ann(Z(R)) \setminus \{0\}| = p^{2r} - 1$. Thus any $Z(R)^*$ such that $x \notin R$ is of degree $p^{2r} - 1$ since every such x is only adjacent to an element in $Ann(Z(R)) \setminus \{0\}$. On the other hand, each element $y \in Ann(Z(R)) \setminus \{0\}$ is adjacent to $Z(R)^* \ni x \notin Ann(Z(R))$. So the degree of y is $p^{3r} - 2$. Now, let $V_1 = \{x \mid Z(R)^* \ni x \notin Ann(Z(R))\}$ and $V_2 = \{y \mid y \in Ann(Z(R)) \setminus \{0\}\}$. Then $|V_2| = p^{2r} - 1$ and $|V_1| = p^{3r} - 1 - (p^{2r} - 1) = p^{3r} - p^{2r}$. Since V_1 and V_2 are mutually disjoint partitions, the automorphism group permutes V_1 and V_2 independently, we obtain $Aut(\Gamma(R_0u_1 \oplus R_0u_2 \oplus R_0v)) \cong S_{p^{3r}-p^{2r}} \times S_{p^{2r}-1}$. Thus,

$$|Aut(\Gamma(R))| = (p^{3r} - p^{2r})!(p^{2r} - 1)!.$$

□

3 Cube Radical Zero Finite Rings of Characteristic p^2 where $p \in Z(R) - (Z(R))^2$

We consider the constructions of the above classes of rings for three different cases as follows:

Case $(i) : pu_1 = pu_2 = pv = 0.$

Let $R_0 = GR(p^{2r}, p^2)$ be a Galois ring of characteristic p^2 and order p^{2r}. Suppose $\{u_1, u_2\}$ and $\{v\}$ are the generating sets for R_0- modules U and V respectively. Then $R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ is and additive abelian group. On this group, define multiplication as: $(a_0, a_1, a_2, a_3)(b_0, b_1, b_2, b_3) = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, a_0b_3 + a_3b_0 + a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2)$. This multiplication turns R into a commutative ring with identity $(1, 0, 0, 0)$.

Proposition 3.1. *Let R be a ring of above construction, the set of zero divisors $Z(R)$ satisfies the following properties:*

(i) $Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$

(ii) $(Z(R))^2 = R_0v$
(iii) $(Z(R))^3 = (0)$

Proof. Let $a \in R_0$ such that $a \notin pR_0$ and $x \in Z(R)$. Then $(a + x)^{p^r} = a^{p^r} + x_1$ where $x_1 \in Z(R)$. But $a^{p^r} + x_1 = a + x_2$, where $x_2 \in Z(R)$. Now, $(a + x_2)^{p^r-1} = 1 + x_3$ where $x_3 \in Z(R)$ and $(1+x_3)^{p^2} = 1$. So $(((a+x)^{p^r})^{p^r-1})^{p^2} = 1$ which shows that $a+x$ is invertible. Further, $|Z(R)| = p^{4r}$ and $|(R_0/pR_0) + Z(R)| = (p^r - 1)p^{4r}$ so that $(R_0/pR_0)^* + Z(R) = R - Z(R)$ which shows that all the elements which lie outside $Z(R)$ are invertible.

(ii) From the multiplication defined on R, consider $pr_0+r_1u_1+r_2u_2+r_3v$ and $pr_0+s_1u_1+s_2u_2+s_3v$ in $Z(R)$. Then $(pr_0+r_1u_1+r_2u_2+r_3v)(pr_0+s_1u_1+s_2u_2+s_3v) = r_1s_1u_1^2+r_1s_1u_1u_2+r_2s_1u_2u_1+r_2s_2u_2^2 \in R_0v$ since $u_1^2 = u_1u_2 = u_2u_1 = u_2^2 = v$ and $v^2 = 0$. Adding the product finitely, we obtain $(Z(R))^2 \subseteq R_0v \cdots\cdots (*)$

Conversely, let $x \in R_0v$, then $x = yv$ where $x \in R_0$, $y \in ann(Z(R))$. and $v \in Z(R)$. From the above argument, there exist $u_1, u_2 \in Z(R)$ such that $v = u_1u_2$. So, $yu_1u_2 \in Z(R).Z(R) = (Z(R))^2$. Thus $x \in (Z(R))^2 \Rightarrow R_0v \subseteq (Z(R))^2 \cdots\cdots (**)$ from $(*)$ and $(**), (Z(R))^2 = R_0v$.

(iii) The product $Z(R)(Z(R))^2 = (Z(R))^2Z(R) = (0)$ since $Z(R))^2 \subseteq Ann(Z(R)) = \{pr_0 + r_1u_1 + r_2u_2 + r_3v \mid r_1 + r_2 \equiv 0(\text{mod } p)\}$ since $RZ(R) = Z(R)R = Z(R)$, the set $Z(R)$ is an ideal. Its uniqueness and maximality follows from the fact that any other ideal distinct from $Z(R)$ contains a unit and is therefore the whole ring R. □

Proposition 3.2. *Let R be a ring of construction in this section. Then*

(i) $|V(\Gamma(R))| = p^{4r} - 1$
(iii) $\Gamma(R)$ *is incomplete*
(iii) $diam(\Gamma(R)) = 2$

Proof. (i) $|V(\Gamma(R))| = |Z(R)^*|$ since $Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$, then $|Z(R)^*| = p^{4r} - 1$. The proofs of (ii), and (iii) follow from the proof of the previous proposition. □

Proposition 3.3. *Let* $R_0 = GR(p^{2r}, p^2)$ *and* R *is a ring constructed in this section with* $pu_1 = pu_2 = pv = 0$. *Then* $Aut(\Gamma(R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v)) \cong S_{p^{4r}-p^{3r}} \times S_{p^{3r}-1}$.

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0/pR_0 \cong \mathbb{F}_p$ form a basis for $\mathbb{F}_p$ over its prime subfield. From the given multiplication, $Ann(Z(R)) = \{pr_0 + a\xi_iu_1 + b\xi_iu_2 + c\xi_iv \mid a+b \equiv (\text{mod } p)\}$. So $|Ann(Z(R))\backslash\{0\}| = p^{3r}-1$. Thus any $Z(R)^* \ni x \notin Ann(Z(R))$ is of degree $p^{3r}-1$ since x is only adjacent to $y \in Ann(Z(R))\backslash\{0\}$. On the other hand, each $y \in Ann(Z(R))\backslash\{0\}$ is adjacent to $Z(R)^* \ni x \notin Ann(Z(R).)$ So, the degree of y is $p^{4r}-2$. Now, let $V_1 = \{x \mid Z(R)^* \ni \notin Ann(Z(R))\}$ and $V_2 = \{y \mid y \in Ann(Z(R)) \setminus \{0\}\}$. Then $|V_1| = p^{3r} - 1$ and $|V_2| = p^{4r} - 1 - (p^{3r} - 1) = p^{4r} - p^{3r}$. Since the automorphisms of the graph permute V_1 and V_2 independently, we obtain $Aut(\Gamma(R)) \cong S_{p^{4r}-p^{3r}} \times S_{p^{3r}-1}$. Consequently, $|Aut(\Gamma(R))| = (p^{4r} - p^{3r})!(p^{3r} - 1)!$. □

Case $(ii) : pu_1 \neq 0,\ pu_2 = 0,\ pv = 0$

Under this case, the set of the zero divisors $Z(R)$ satisfies the following properties:

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v,$$

$$(Z(R))^2 = R_0u_1 \oplus R_0v,$$

$$(Z(R))^3 = (0).$$

Considering $(pr_0, r_1, r_2, r_3), (ps_0, s_1, s_2, s_3) \in Z(R)$. Then, $(pr_0, r_1, r_2, r_3)(ps_0, s_1, s_2, s_3) = (0, pr_0s_1 + ps_0r_1, 0, r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2) \in R_0u_1 \oplus R_0v$. The rest of the steps are similar to the ones in case (i) giving rise to the following in the sequel:

Proposition 3.4. *Let* R *be a ring constructed in this section, with* $pu_1 \neq 0,\ pu_2 = pv = 0$. *Then,*

(i) $|V(\Gamma(R))| = p^{5r} - 1$
(ii) $diam(\Gamma(R)) = 2$
(iii) $gr(\Gamma(R)) = 3$

Next, we determine the structure of the automorphisms of the graph of the ring considered under case (ii).

Proposition 3.5. *Let* $R_0 = GR(p^{2r}, p^2)$ *and* R *be a ring of the construction with* $pu_1 \neq 0, pu_2 = pv = 0$. *Then,* $Aut(\Gamma(R)) \cong S_{p^{2r}-1} \times S_{p^{5r}-p^{4r}-p^{2r}} \times S_{p^{3r}} \times S_{p^{3r}}$.

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0/pR_0 \cong \mathbb{F}_p$ form a basis for $\mathbb{F}_p$ over its prime subfield. The annihilator $Ann(Z(R)) = \{p\xi_iu_1 + c\xi_iv \mid c \in R_0\}$. Now, $|Ann(Z(R))| = p^{2r}$ and $|Ann(Z(R)) \setminus \{0\}| = p^{2r} - 1$. Thus any $Z(R)^* \ni x \in Ann(Z(R))$ is of degree $p^{5r} - 2$, since x is adjacent to every $y \in Z(R)^*$.

Let $V_1 = (Ann(Z(R)))^*$

$V_2 = \{pr_0 + a\xi_iu_1 + b\xi_iu_2 + c\xi_iv \mid a + b \equiv (\text{mod } p)\} - Ann(Z(R))$.

Now, $|V_2| = p^r(p^{3r} - p^{2r})p^r - p^{2r} = (p^{4r} - p^{3r})p^r - p^{2r} = p^{5r} - p^{4r} - p^{2r}$.

Each vertex in V_2 is adjacent to an element of the form $pr_0 + a'\xi_iu_1 + b'\xi_iu_2 + c\xi_i$ where $pa' + ap = 0$. So the degree of each vertex in V_2 is $p^{4r} - 2$.

Consider $V_3 = \{pr_0 + \xi_i u_1 + c\xi_i v \mid c \in R_0\}$. Then $|V_3| = p^{3r}$. Each vertex in V_3 is adjacent to an element of the form $\xi_i u_1 + \xi_i u_2 + cv$ or $p\xi_i u_1 + c\xi_i v$. So the degree of each vertex in V_3 is $p^{2r} + p^{2r} - 1 = 2p^{2r} - 1$.

Finally, $V_4 = \{pr_0 + p\xi_i u_1 + \xi_i u_2 + c\xi_i v \mid c \in R_0\}$, so that $|V_4| = p^{3r}$. Each vertex of V_4 is adjacent to a vertex of V_2 or V_1. Therefore the degree of a vertex in V_4 is $p^{5r} - p^{4r} - p^{2r} + p^{2r} - 1 = p^{5r} - p^{4r} - 1$. Since the automorphisms permute V_1, V_2, V_3 and V_4 independently, the results easily follows. □

Case $(iii) : pu_1 \neq 0,\ pu_2 \neq 0,\ pv = 0$

In this case, the inherent properties of the multiplication together with the fact that $pu_1 \neq 0,\ pu_2 \neq 0$ gives a characterization of the structures of the zero divisors as:

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v,$$

$$(Z(R))^2 = R_0u_1 \oplus R_0u_2 \oplus R_0v,$$
$$(Z(R))^3 = (0).$$

Let $(pr_0, r_1, r_2, r_3), (ps_0, s_1, s_2, s_3) \in Z(R)$. Then

$(pr_0, r_1, r_2, r_3)(ps_0, s_1, s_2, s_3) = (0, pr_0s_1 + ps_0r_1, pr_0s_2 + ps_0r_2, pr_0s_3 + ps_0r_3 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2) \in R_0u_1 \oplus R_0u_2 \oplus R_0v$. Since $Z(R)$ is an ideal, $(Z(R))^2 Z(R) = Z(R)(Z(R))^2 = (0)$.

The following results summarizes some properties of zero divisor graph of the ring constructed in this section.

Proposition 3.6. *Let R be a ring constructed in this section, with $pu_1 \neq 0,\ pu_2 \neq 0,\ pv = 0$. Then*

(i) $|V(\Gamma(R))| = p^{6r} - 1$
(ii) $diam(\Gamma(R)) = 2$
(iii) $gr(\Gamma(R)) = 3$

The structure of the automorphisms of the graph of the ring considered in this section is summarized in the following result.

Proposition 3.7. *Let $R_0 = GR(p^{2r}, p^2)$ and R is the ring constructed in this section with $pu_1 \neq 0,\ pu_2 \neq 0,\ pv = 0$. Then,*

$$Aut(\Gamma(R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v)) \cong S_{p^{3r}-1} \times S_{2p^{2r}} \times S_{2(p^{3r}-p^{2r})} \times S_{2p^{5r}-2p^{4r}-2p^{3r}}.$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0/pR_0 \cong \mathbb{F}_p$ form a basis for $\mathbb{F}_p$ over its prime subfield. Consider $V_1 = Ann(Z(R))^* = \{p\xi_i u_1 + p\xi_i u_2 + c\xi_i v \mid c \in R_0\}$. Then $|V_1| = p^{3r} - 1$ and each $v \in V_1$ is adjacent to every other vertex in the graph. Therefore $degv = p^{6r} - 2$ for all $v \in V_1$. Let $V_2 = \{\xi_i u_1 + c\xi_i v\} \cup \{\xi_i u_2 + c\xi_i v\}$ where $C \in R_0$. Then $|V_2| = 2p^{2r}$. Now, let $A = \{pr_0 + \xi_i u_1 + c\xi_i v\} \cup \{pr_0 + \xi_i u_2 + c\xi_i v\}$. Consider $V_3 = A - V_2$. Then $|V_3| = 2(p^{3r} - p^{2r})$. Finally, we assume $B = \{pr_0 + a\xi_i u_1 + b\xi_i u_2 + c\xi_i v \mid a + b \equiv (\text{mod } p)\}$, and consider $V_4 = B - Ann(Z(R))$. Then $|V_4| = 2[p^r(p^{3r} - p^{2r})p^r - p^{3r}] = (2p^{4r} - 2p^{3r})p^r - 2p^{3r} = 2p^{5r} - 2p^{4r} - 2p^{3r}$. Using the fact that automorphisms permute V_1, V_2, V_3 and V_4 independently, we obtain the result. □

4 Cube Radical Zero Finite Rings of $CharR = p^2$ where $p^2 \in Z(R)$

In this case, the product of any two of the elements of R is given as follows:

$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0 + pr_1s_1, r_0s_1 + r_1s_0, r_0s_2 + r_2s_0, r_0s_3 + r_3s_0 + r_1s_1 + r_1s_2 + r_2s_1 +$

r_2s_2) and $pu_1 = pu_2 = pv = 0$, $u_1^2 = p\xi_i a + b\xi_i v$, $u_2^2 = c\xi_i v$ and $v^2 = 0, a, b, c \in R_0$. The structure of the zero divisors is given by $Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ so that $(Z(R))^2 = pR_0 \oplus R_0v$ and $(Z(R))^3 = (0)$.

Proposition 4.1. *Let $R_0 = GR(p^{2r}, p^2)$ and $R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ is a ring with respect to the multiplication in this section, with $p^2 \in Z(R)$, $v^2 = pu_1 = pu_2 = pv = 0, u_1^2 = p\xi_i a + b\xi_i v$, $u_2^2 = c\xi_i v$, $a, b, c \in R_0$. Then,*

$$Aut(\Gamma(R)) \cong S_{p^{2r}-1} \times S_{p^{3r}-p^{2r}} \times S_{p^{4r}-p^{3r}}.$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0/pR_0 \cong \mathbb{F}_p$ form a basis for $\mathbb{F}_p$ over its prime subfield. From the given multiplication, $Ann(Z(R)) = \{pr_0 + c\xi_i v \mid r_0, c \in R_0\}$. Consider $V_1 = Ann(Z(R)) \setminus \{0\}$. Then $V_1 = p^{2r} - 1$ and each $x \in V_1$ is adjacent to every other vertex in the graph. So $deg(x) = p^{4r} - 2$. Let $V_2 = \{pr_0 + \xi_i u_1 + c\xi_j v \mid r_0, c \in R_0\}$. Then $|V_2| = p^{3r} - p^2$ and each vertex $y \in V_2$ is adjacent to the vertices in V_1. So the degree of a vertex in V_2 is $p^{3r} - p^{2r} - 1$. Finally, the vertex set $V_3 = \{pr_0 + a\xi_i u_1 + b\xi_j u_2 + c\xi_k v \mid a + b \equiv (\bmod\ p), c \in R_0\} \cup \{pr_0 + \xi_i u_2 + d\xi_i v \mid d \in R_0\}$, so that $|V_3| = p^{4r} - p^{3r}$ and each $z \in V_3$ is adjacent to the other vertices in V_1 or each vertex of the form $pr_0 + a\xi_i u_1 + b\xi_j u_2 + c\xi_k v$, $\mid a + b \equiv (\bmod\ p)$ and vice versa. So the degree of the vertex in V_3 is $p^{3r} - p^{2r} + p^{2r} - 1 = p^{3r}$. The result easily follow from the fact that automorphisms permute V_1, V_2 and V_3 independently. □

5 Cube Radical Zero Finite Rings of Characteristic p^3

The product of the elements is given as follows:

$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0, r_0s_1 + r_1s_0, r_0s_2 + r_2s_0, r_0s_3 + r_3s_0 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2)$. Using the given multiplication, the zero divisors satisfy the following properties:

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v,$$
$$(Z(R))^2 = p^2R_0 \oplus R_0v,$$
$$(Z(R))^3 = (0).$$

Proposition 5.1. *Let $R_0 = GR(p^{3r}, p^3)$ and $R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ is a ring with respect to the multiplication given in this section, with $pu_1 = pu_2 = pv = 0$; $u_1^2 = p^2\xi_i a + b\xi_j v$, $u_2^2 = c\xi_i v$, $v^2 = 0, a, b, c \in R_0$. Then,*

$$Aut(\Gamma(R)) \cong S_{p^{4r}-p^{3r}-1} \times S_{p^{3r}} \times S_{p^{5r}-2p^{4r}+p^{3r}} \times S_{p^{4r}-p^{3r}}.$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0/pR_0 \cong \mathbb{F}_p$ form a basis for $\mathbb{F}_p$ over its prime subfield. Using the given multiplication, $Ann(Z(R)) = \{p^2r_0 + a\xi_i u_1 + b\xi_i u_2 + c\xi_i v \mid a, b, c \in R_0, a + b \equiv 0(\bmod\ p)\}$. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. Then $|V_1| = p^r(p^{2r} - p^r)p^r - 1 = p^{4r} - p^{3r} - 1$. Since each vertex in V_1 is adjacent to every other vertex in the graph, the degree of $x \in V_1$ is $p^{5r} - 2$.

Next, consider $V_2 = \{pr_0 + a\xi_i u_1 + b\xi_i u_2 + c\xi_i v \mid a, b, c \in R_0, a + b \equiv 0(\bmod\ p)\} \setminus V_1$. Then $|V_2| = (p^{2r} - p^r)(p^{2r} - p^r)p^r = (p^{2r} - p^r)(p^{3r} - p^r) = p^{5r} - 2p^{4r} + p^{3r}$. Every vertex $y \in V_2$ is adjacent to a vertex in V_1 or a vertex of the form $p^2r_0 + a'\xi_i u_1 + b'\xi_i u_2 + c'\xi_i + c'\xi_i v$, where $a', b', c' \in R_0$, $a' + b' \not\equiv 0(\bmod\ p)$. So the $deg(y) = p^{4r} - p^{3r} - 1 + p^{3r} = p^{4r} - 1$. Let $V_3 = \{p^2r_0 + a'\xi_i u_i + b'\xi_i u_2 + c'\xi_i v \mid a', b', c' \in R_0, a' + b' \not\equiv 0(\bmod\ p)\}$. Then $|V_3| = p^{4r} - 1 - (p^{4r} - p^{3r} - 1) = p^{3r}$. Each vertex $z \in V_3$ is adjacent to a vertex in V_1 or V_2. So $deg(z) = p^{5r} - 2p^{4r} + p^{3r} + p^{4r} - p^{3r} - 1 = p^{5r} - p^{4r} - 1$. Finally, $V_4 = \{pr_0 + a'\xi_i u_1 + b'\xi_i u_2 + c'\xi_i v \mid a', b', c' \in R_0, a' + b' \not\equiv 0(\bmod\ p)\} - V_2$. Then $|V_4| = (p^{2r} - p^r)p^r.p^r = p^{4r} - p^{3r}$. Each vertex in V_4 is adjacent to all the vertices in V_1 but neither in V_2 nor in V_3. Therefore, $deg(w) = p^{4r} - p^{3r} - 1$ for each $w \in V_4$. Since automorphisms permute the vertices of V_1, V_2, V_3 and V_4 independently, the results easily follows. □

6 Conclusion

In Study we determined the automorphisms of such rings in which the product of any three zero divisor is zero and revealed the structures and order formulae for automorphisms. This was achieved by partitioning the ring under consideration into mutually disjoint subset of invertible elements and zero divisors, isolation of zero divisors and determination of there graphs using case to case basis discovery of there maps. To this end, research in this area is still minimal and we recommend other researchers to carry out more studies regarding automorphisms of zero divisor graphs in future.

Acknowledgements

This work was supported by University of Kabianga annual research grant.

Competing Interests

Authors have declared no competing interest.

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