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Solution of Third Order Viscous Wave Equation Using Finite Difference Method

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Abstract

The aim of this paper was to solve the third order viscous wave equation: $u_{tt} = vu_{xx} + c^2u_{xxt}$, which is a PDE. It occurs in many real-life situations such as water waves, sound waves, radio waves, light waves and seismic waves. This equation has been solved before using analytical methods but not yet been exhaustively nor conclusively done. Two schemes, namely CD-FD and CN-FD were developed and the equation discretised by FDM. We used each scheme respectively to obtain solution algorithms. Stability of the schemes was analysed, consistency of the numerical solutions with the original equation was tested, and Mathematica software used to generate solutions. The numerical computational results obtained for solutions of third order viscous wave equation obtained for varying the mesh ratio showed that the schemes were both conditionally stable and consistency noted. We found that as the mesh ratio reduces, the solution tends towards the exact solution. The solution algorithm showed consistency with the original viscous equation when tested. In addition, the equation simulates many physical situations which include designing of bridges, acoustics, gas dynamics, seismology, meteorology among many other natural phenomena. This work contributes to mathematical knowledge in research and innovations which apply PDEs.

Keywords: Discretisation, Finite Difference Schemes, Stability and Consistency, Finite Difference Method

1 Introduction

Finite Difference Method has been used to solve a variety of physical problems, [9, 16, 29]. The FDM is becoming increasingly more important in the seismic industry and structural modelling due to its relative accuracy and computational efficiency. It is the most commonly used approach in numerical modelling, [31]. Seismic waves may be simulated by a viscous type of wave equation which is known to be difficult to obtain the analytical solution. This is where the numerical solution to such types of equations is needed to solve practical problems, [20].
Hyperbolic PDEs arise in a broad spectrum of applications where wave motion is involved, examples; optics, acoustics, oil and gas dynamics and vibrating string to name but a few. Waves have distinct properties specific to their type but also exhibit characteristics in common with more abstract waves such as sound waves and light waves [5, 7, 11].
We solved the equation

$$u_{tt} = vu_{xx} + c^2u_{xxt}, t > 0, 0 < x < L, 0 < t < T, -\infty < u < \infty \tag{1.1}$$

where $u = u(x, t)$, denotes a dependent function on two variables, which include spatial x, time t and their partial derivatives, v is the velocity coefficient, c the viscosity coefficient and T is a given time constant. The initial and boundary conditions are given as follows;

$$u(x, 0) = 0 = g(x) = \sin(x_m) : x \in (0, L) \tag{1.2}$$

$$u_t(x, 0) = 0 : x \in (0, L) \tag{1.3}$$

$$u(0, t) = 0, t \in (0, T) \tag{1.4}$$

$$f(t) = sin(t) : u(0, t) = sin(t) \tag{1.5}$$

$$u(x, t) = u(5.5, t) = 0 : t > 0 \tag{1.6}$$

$$u(L, t) = 0, t \in (0, T) \tag{1.7}$$

2 Approximations of Derivatives

We used Taylor's series expansion of $u(x+h, t)$ and $u(x-h, t)$ about a selected grid point $U_{m,n}$ as according to, [17, 21, 26]. To start with, take the function and its partial derivatives to be continuous. The finite difference approximations to derivatives was then obtained from the expansion of two variables x and t. When we truncated the terms of $O(h^2)$ and above on the expansion and re-arranged we obtained;

$$u_{xx} \approx \frac{1}{h^2}[u(x + h, t) - 2u(x, t) + u(x - h, t)]; O(h^2) \tag{2.1}$$

This equation is the Centred Difference (CD) approximation to u_{xx} with an error of order h^2 .

$$u_t \approx \frac{1}{h}[u(x,t+k) - u(x,t)];\ O(k) \tag{2.2}$$

which is the Forward Difference (FD) approximation to u_t with an error of order k (first order accurate). Similarly, truncating the terms of $O(h^2)$ and above and re-arranging yielded;

$$u_{tt} \approx \frac{1}{k^2}[u(x,t+k) - 2u(x,t) + u(x,t-h)]; O(k^2) \tag{2.3}$$

This equation is the Centred Difference (CD) approximation to u_{tt} with an error of order k^2 . Equations (2.1), (2.2) and (2.3) respectively were re-written and stated with numerical symbols thus;

$$u_{xx} = \frac{1}{h^2}[U_{m+1,n} - 2U_{m,n} + U_{m-1,n}]; O(h^2) \tag{2.4}$$

Similarly, we had the approximations for the Partial Derivatives with respect to t as listed below;

$$u_t = \frac{1}{k}[U_{m,n+1} - U_{m,n}]; O(k) \tag{2.5}$$

this was referred to as the FD in time, t

$$u_{tt} = \frac{1}{k^2}[U_{m,n+1} - 2U_{m,n} + U_{m,n-1}]; O(k^2) \tag{2.6}$$

Finally, this was referred to as the CD scheme of second order derivative with respect to time, t. All these are finite difference schemes of either first or second order accuracies in space and time, [8, 9]. Let $U(x,t) = U_{(m+h,n+k)}$ with $U_{m,n}$ as the point of reference. The values of h and k was then varied iteratively and in steps such as h=0,1, 2 and so on. In numerical terms, this is written as $U_{m+1,n+1}, U_{m,n+1}, U_{m+1,n+2}$ and may continue in the same manner.

3 Finite Difference Method

This section dealt with the discretisation of the equation (1.1) by FDM. We developed, analysed the stability and consistency of the numerical algorithm which may also be referred to as a numerical solution. The method forms the basis for most of the numerical solutions without alterations to the original equation. Numerical solution schemes to equations are used in many forms in numerical solvers. An advantage of this method is how it can be

applied directly to the differential form of the governing equations. The principle here is to employ a Taylor's series expansion for the approximation of the derivatives concerned. Another important advantage is its simplicity. It provides the possibility to easily obtain high-order approximations and hence to achieve high-order accuracy of the spatial discretisation. The FDM is the most straight forward numerical approach in seismic modelling, and it is also becoming increasingly more important in the seismic industry and structural modelling due to its relative accuracy and computational efficiency, [31].
On the other hand, because the method requires a structured grid, the range of application should be clearly restricted, which is a disadvantage. However, with this method, the governing equation has to first be transformed into a Cartesian coordinate system or in other words from the physical to the computational space. Nowadays, this method is utilized in the research of turbulent flows together with immersed boundary cells, in biology. Sometimes PDEs are very difficult to solve analytically or similarly when models are needed for computer simulations, [17]. In these cases, at this juncture, FDM is used to solve the equations instead of analytical approach. This method is also very good for solving eigenvalue problems which is an added advantage. Many real life problems generally do not have analytical solutions and this is where numerical techniques come in handy, [9, 3, 21].

4 Numerical schemes

We considered the equation $u_{tt} = vu_{xx} + c^2 u_{xxt}$, it is known to be a hyperbolic PDE. Here the second partial derivative with respect to time, u_{tt}, was approximated by equation (2.6). Likewise the derivative u_{xx} is of second order accurate in space was approximated by equation(2.4) and u_t which is first order accurate in in time by equation (2.5). While the mixed third order derivative was obtained by first replacing the partial derivatives in turns. Centred-forward-approximation, (CD-FD), was used to discretise u_{xxt}. To achieve this, we used the centred- difference approximation for u_{xx} and a forward difference approximation for partial derivative in time, t, u_t. We started with the viscosity term stated as; $u_{xxt} = \frac{\partial}{\partial t}(u_{xx})$ where numerical second spatial derivative in x was as shown in equation (2.4) and the first derivative in time t, in equation (2.5). Therefore it follows that the third order term becomes;

$$u_{xxt} = \frac{1}{k}[(u_{xx})_{m,n+1} - (u_{xx})_{m,n}]; O(h^2) + O(k) \tag{4.1}$$

simplified, ignored higher order terms and simplified further, for the right hand side of the equation and we got;

$$\frac{1}{kh^2}[U_{m+1,n+1}+U_{m-1,n+1}-((U_{m+1,n}+U_{m-1,n})+2(U_{m,n}-U_{m,n+1})];O(h^2)+O(k) \quad (4.2)$$

This equation has been obtained using the already truncated approximations for u_{xx} and u_t, [7].

4.1 Centred-Forward Difference Scheme

We replaced u_{tt} in equation (1.1) by the equation (2.6), u_{xx} by the equation (2.4). On the other hand, to discretise the third order viscous term, u_{xxt}, we combined u_{xx} and u_t using the already defined derivatives as discussed below. The FDM allows replacements term by term using numerical schemes and we obtained an algebraic equation as stated in equation (4.3) below;
First, let the mesh ratio $r = \frac{k^2}{h^2}v$, and $\beta = \frac{k}{h^2}c^2$, then the original (1.1) becomes;

$$\beta U_{m-1,n+1} - (2\beta+1)U_{m,n+1} +_{m+1,n+1}$$
$$= (r-\beta)U_{m-1,n} + (2\beta+2-2r)U_{m,n} + (r-\beta)U_{m+1,n} - U_{m,n-1}; O(h^2)+O(k^2) \quad (4.3)$$

Where $m = 1,2,3,\dots(E-1), E$ and E was number of divisions along the x-axis. The final scheme (4.3) obtained was referred to as CD-FD numerical scheme and known as a numerical solution to the third order wave equation by CD-FD scheme.

4.2 Stability of Centred-Forward Difference Scheme

The stability of a finite difference solution is an important characteristic and must be established before getting into solving any equation. This concerns the nature of errors generated during discretisation and accumulated over the previously approximated schemes, [23]. The actual solution to the difference equation can be found if only we could possibly carry out all numerical operations extended to an infinite number of decimal places. However, every calculation made by the computer is carried out to finite number of significant figures. As a result, this will bring out a round off error at each level, [2, 22, 11, 28].
CD-FD scheme was expanded iteratively by taking m=1,2,3,...(E-1) obtained a system of equations;

$$-\beta U_{E-2,n+1} + (2\beta+1)U_{E-1,n+1} - \beta U_{E+1,n+1} = (r-\beta)U_0, n + (2\beta-2r+2)U_{E-1,n} +$$
$$(r-\beta)U_{E,n} - U_{E+1,n-1} \quad (4.4)$$

which are linear algebraic form of equation (4.3). It was re-written in matrix-vector form and stability analysis was done using matrix-eigen value method. It was found that the CD-FD scheme was conditionally stable.

4.3 Crank Nicolson scheme

Next we replaced u_{tt} by its CD-FD scheme, u_{xx} by the average of its centred difference approximations at n^{th} time level and at $(n+1)^{th}$ time level. We referred to this as CN-FD scheme, which yielded equation (4.5) as shown below; To start with, let $r = \frac{k^2}{h^2}v$ and $\beta = \frac{k}{h^2}c^2$, the scheme (4.4) becomes;

$$\begin{aligned} -(2\beta + r)U_{m-1,n+1} + (4\beta + 2r + 2)U_{m,n+1} - (2\beta + r)U_{m+1,n+1} \\ = (r - 2\beta)U_{m-1,n} + (4\beta + 4 - 2r)U_{m,n} + (r - 2\beta)U_{m+1,n} - 2U_{m,n-1} \ldots \end{aligned} \tag{4.5}$$

This is for $m = 1, 2, 3, \ldots, (E-1), E$ where E is number of divisions along the x-axis. The final scheme obtained is referred to as CN-FD numerical scheme.

4.4 Stability Analysis of Crank-Nicolson Scheme

We used the matrix method to analyse stability of the scheme (4.5). Expanding this scheme by taking $m = 1, 2, 3, (E-1)$ and obtained the following general result of system of equations.

$$\begin{aligned} (2\beta + r)U_{E-2,n+1} + (4\beta + 2r + 2)U_{E-1,n+1} - (2\beta + r)U_{E,n+1} \\ = (r - 2\beta)U_{E-2,n} + (4\beta + 4 - 2r)U_{E-1,n} + (r - 2\beta)U_{E,n} - 2U_{E+1,n-1} \end{aligned} \tag{4.6}$$

the systems of linear algebraic of equation (4.5) was written in matrix-vector form as and stability Analysis was done using Matrix Method. It was found that the CN-FD scheme was also conditionally stable.

5 Consistency of the whole Equation

Truncations carried forward accumulated errors, therefore, it was necessary to find out the consistency of the numerical solution with the original equation. We used a related concept of error analysis. The process of testing consistency was done step by step using the application of Taylors expansion. The viscous term being u_{xxt}. This is the term which determines the order of the model equation. The other two terms are both second order, one in space, x and the other in time, t. We carried out expansion of equation (1.1) using Taylors expansion. It was found that the CD-FD and CN-FD numerical solution algorithms were both consistent with the original equation, [12].

6 Results

From the initial and boundary conditions, $u_t(x, 0) = 0$, the forward difference analogue of u_t yields;

$u_t \approx \frac{1}{k}[U_{m,n+1} - U_{m,0}] = 0$, that is $\frac{1}{k}[U_{m,1} - U_{m,0}] = 0$
implying that $U_{m,1} = U_{m,0}$.
Similarly, from the initial condition, $u(x,0) = 0$, $u(x,0) \approx U_{m,0} = sin(x_m)$.
Therefore $U_{m,0} = 0$ and $U_{m,1} = 0$

7 Graphical Presentations

7.1 Case one

We use CD-FD scheme and Mathematica software, to find the solution of the third order viscous wave equation at two levels, namely n = 1 , 2 to start with. An appropriate temporal mesh size considered for the calculation is taken for different values of h, k, r, β and t,we got the results as follows for the given values thus; $h = \frac{1}{2}$, $k = \frac{1}{2}$, $\beta = 2, r = 1$, the schemes become;

CD-FD :

$$-2U_{m-1,n+1}+5U_{m,n+1}-2U_{m+1,n+1} = -U_{m-1,n}+4U_{m,n}-U_{m+1,n}-U_{m,n-1} \quad (7.1)$$

CN-FD :

$$-5U_{m-1,n+1}+12U_{m,n+1}-5U_{m+1,n+1} = -3U_{m-1,n}+10U_{m,n}-3U_{m+1,n}-2U_{m,n-1} \quad (7.2)$$

The solutions at $t = 0$ and$t = 1$ are already known and are referred to as the zero and the first time level solutions respectively for $0 \leq x \leq 5$. Using the initial and boundary conditions stated as equation (1.1) earlier,
$u(0,t) = f(t), u(5,t) = 0 : t > 0$, let $f(t) = sin(t)$ implying that $u(0,t) = sin(t)$. Also let $u(\infty,t) = u(5.5,t) = 0 : t > 0$, $u(x,0) = 0$, $u_t(x,0) = 0$.
Here, the subscript m - designates the grid point along the x - direction, and n along the t - direction as described earlier. Therefore n is varied as m =1, 2, 3....10; with m and n fixed as n=0 and m = 1,2 3, ...10 in Equation (3.7) repeatedly. The systems of linear algebraic equations obtained are by fixing n (from $n = 1$) and vary m and the mesh ratio, tabulate values and present graphically for both case one and two.

7.1.1 Forward Difference Scheme

We use Mathematica to output the results graphically
In[11]:ListPlot3D
[0,0.4794255386,0.8414709848,0.9974949866,0.9092974268,
0,0.300879,0.589256,0.719299,0.643848,0,0.150439,0.369843,0.544545,0.594099
0,0.0752186,0.222522,0.383488,0.488624,0,0.0376076,0.130048,0.256684,0.372362,

0,0.0188004,0.074393,0.165388,0.268373,0,0.00939329,0.0418337,0.103342,0.185006,
0,0.00468287,0.0231464,0.0627647,0.12245,0,0.00231389,0.0125201,0.0367957,0.0772317,
0,0.00110185,0.00641843,0.0201236,0.0447421,0,0.000440741,0.00269959,0.00883728,
0.020402,0,0,0,0,0,
PlotLabel → " Graphical Presentation with FDS ",AxesLabel→ "t values","xvalues","u(x,t) values",Shading → True]
Out[1] : - Surface Graphics -

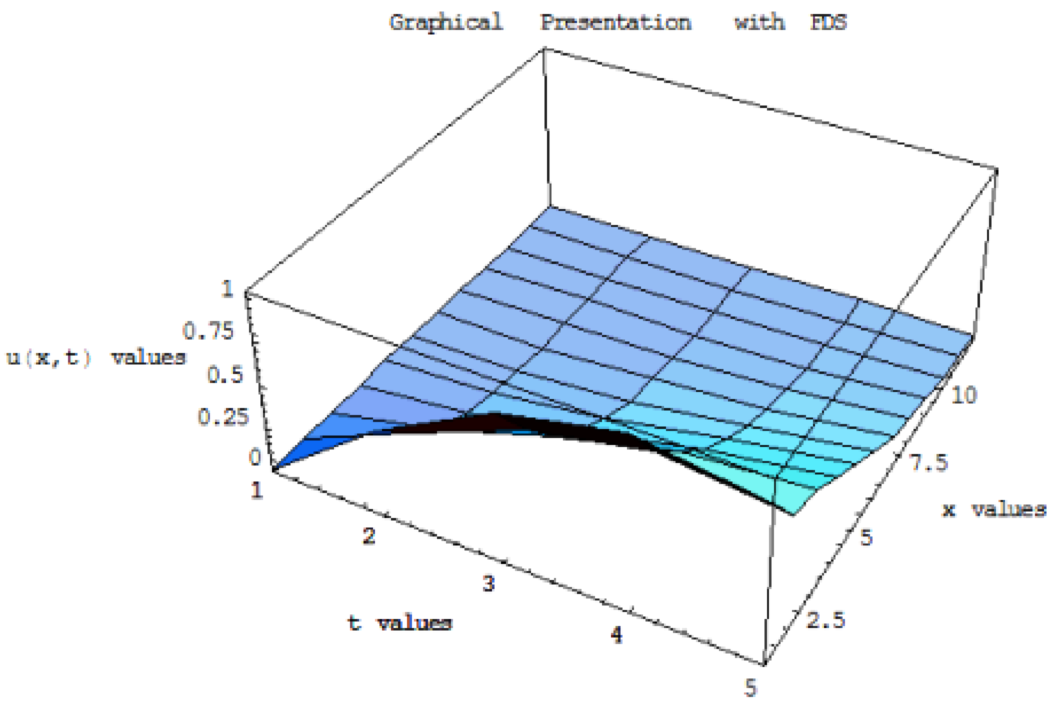

Figure 1: Graphical Presentation Using FDS with $\beta = 2$ and r=1

7.1.2 Crank Nicolson Scheme

We use mathematica to output the results graphically as shown below.
In[11]:ListPlot3D [0,0.4794255386,0.8414709848,0.9974949866,0.9092974268,
0,0.297218,0.568157,0.679295,0.590844,0,0.159508,0.372232,0.525419,0.549056,
0,0.0856011,0.235876,0.383471,0.462605,0,0.0459346,0.145934,0.268298,0.364835,
0,0.0246419,0.0886415,0.181662,0.273855,0,0.013206,0.0530065,0.119628,0.197307,
0,0.00705256,0.0311787,0.0766073,0.136589,0,0.0037201,0.017873,0.0472206,0.089872,
0,0.00187568,0.009633
PlotLabel→ " Graphical Presentation with CNS ",AxesLabel→"t values","xvalues","u(x,t) values",Shading → True]
Out [2]: -Surface Graphics-

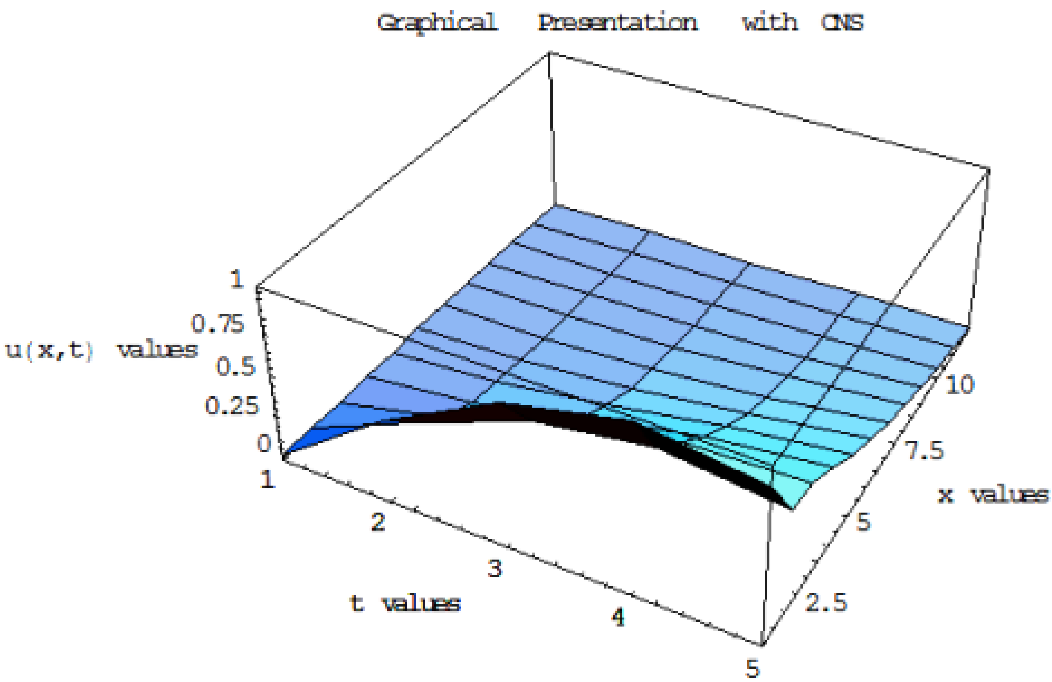

Figure 2: Graphical Presentation Using CNS with $\beta = 2$ and r=1

8 Graphical Presentations

8.1 Case Two

For $h = 1/5, k = 1/5, \ = 5, r = 1$, the schemes become;

CD-FD:

$$-5U_{m-1,n+1} + 11U_{m,n+1} - 5U_{m+1,n+1} = 4 - U_{m-1,n} + 10U_{m,n} - 4U_{m+1,n} - U_{m,n-1} \tag{8.1}$$

CN-FD:

$$-11U_{m-1,n+1} + 24U_{m,n+1} - 11U_{m+1,n+1} = -9U_{m-1,n} + 22U_{m,n} - 9U_{m+1,n} - 2U_{m,n-1} \tag{8.2}$$

The solutions at $t = 0$, $t = 1$ are already known For $0 \leq x \leq 5$. From the initial and boundary conditions, $u(0,t) = f(t), u(\infty, t), t) = 0 : t > 0$
Let $f(t) = sin(t)$ implying that $u(0,t) = sin(t)$
Let $u(\infty, t), t) = u(5.2, t) = 0 : t > 0$

$u(x, 0) = 0$
$u_t(x, 0) = 0$
We fix n (from n=1) and vary m iteratively to obtain systems of equations

8.1.1 Forward Difference Scheme

In[12]:ListPlot3D
[0, 0.1986693308,0.3894183423,0.5646424734,0.7173560909,
0,0.147911,0.319491 ,0.48469,0.630683,0,0.0949205,0.229887,0.375187,0.514584,
0,0.0609145,0.163479,0.286306,0.413802,0,0.0390914,0.115148,0.21594,0.328767,
0,0.0250866,0.0804646,0.161439,0.25883,0,0.0160992,0.0558534,0.120115,0.202718,
0,0.0103315,0.0385489,0.0895383,0.158872,0,0.00663017,0.0264746,0.0676999,
0.125685,0,0.00425486,0.0181041,0.0531007,0.101642,0,0.00273052,0.0123331,
0.0448106,0.0853554, 0,0.00175229,0.00837354,0.0425446,0.0755013,0,0.00112452,
0.0056681,0.0288309,0.0562462, 0,0.000721646,0.00382638,0.019532,0.0413734,0,
0.000463104,0.00257675,0.013227,0.0301229, 0,0.000297184,0.00173132,0.00895258,
0.0217459,0,0.0001907,0.00116083,0.00605559,0.0155854, 0,0.000122355,0.000776737,
0.00409265,0.011099, 0,0.0000784825,0.000518628,0.0027628,0.00785728, 0,0.000050306,
0.000345408,0.00186166,0.00552855, 0,0.0000321906,0.000229197,0.00125036,0.00386182,
0,0.0000205135,0.000151099,0.000834386,0.00266941, 0,0.0000129389,0.000098298,
0.000549187,0.00181197, 0,0.00000795223,0.0000620512,0.000350337,0.00118598,
0,
0.00000455597,0.000036306,0.000206727,0.000713659, 0,0.00000297089,0.0000167287,
0.0000957841,0.000334773,0,0,0,0,0,
PlotLabel→" Graphical Presentation with FDS ",AxesLabel→"tvalues","x values","u(x,t) values",Shading→ True]
Out[3]:-Surface Graphics-

8.1.2 Crank Nicolson Scheme

In[12]:ListPlot3D[0, 0.1986693308,0.3894183423,0.5646424734,0.7173560909,
0,0.148583,0.319312,0.482989,0.627138, 0,0.0973107,0.233105,0.377532,0.514727,
0,0.0637312,0.168371,0.291213,0.416658, 0,0.0417391,0.120556,0.222047,0.333098,
0,0.027336,0.0856915,0.167599,0.263317, 0,0.017903,0.0605332,0.125375,0.20605,
0,0.0117251,0.0425341,0.093047,0.159757, 0,0.00767905,0.029749,0.0685674,0.122831,
0,0.0050292,0.0207227,0.0502084,0.093721, 0,0.00329374,0.0143836,0.0365551,
0.0710113, 0,0.00215715,0.00995195,0.0264773,0.0534606, 0,0.00141276,0.0068662,
0.0190881,0.0400111, 0,0.000925244,0.00472529,0.0137026,0.0297829, 0,0.000605952,
0.00324471,0.00979862,0.0220584, 0,0.000396832,0.00222387,0.00698267,0.0162615,
0,
0.000259864,0.0015221,0.00496082,0.0119362, 0,0.000170143,0.00104124,0.00351544,

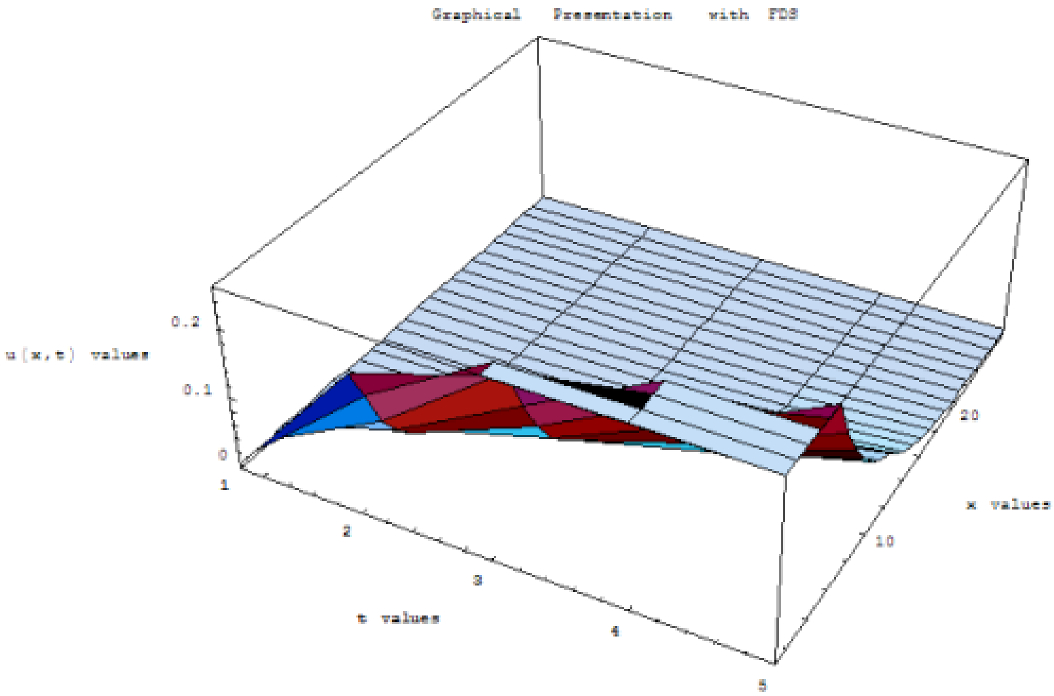

Figure 3: Graphical Presentation Using FDS with $\beta = 5$ and r=1

0.00872554, 0,0.000111358,0.000713128,0.00248651,0.00635306, 0,0.0000728198, 0.000490753,0.00175712,0.00460624, 0,0.0000475214,0.000341958,0.00124218,0.00332264, 0,0.0000308633,0.000245126,0.000879909,0.00237855, 0,0.0000198166,0.00018623, 0.000625131,0.00167946,0,0.000012373,0.000118614,0.000412762,0.00112698, 0, 0.00000717908,0.0000699052,0.000250027,0.000691476, 0,0.00000329041,0.0000323639, 0.000117774,0.000328429,0,0,0,0,0,
PlotLabel →" Graphical Presentation with CNS ",AxesLabel→"t values","xvalues","u(x,t) values",Shading→ True]
Out[4]: -Surface Graphics-

From the graphics, the following can be interpreted: The surface of the plot is not smooth because the differential equation is satisfied only at a selected number of discrete nodes within the region of integration.

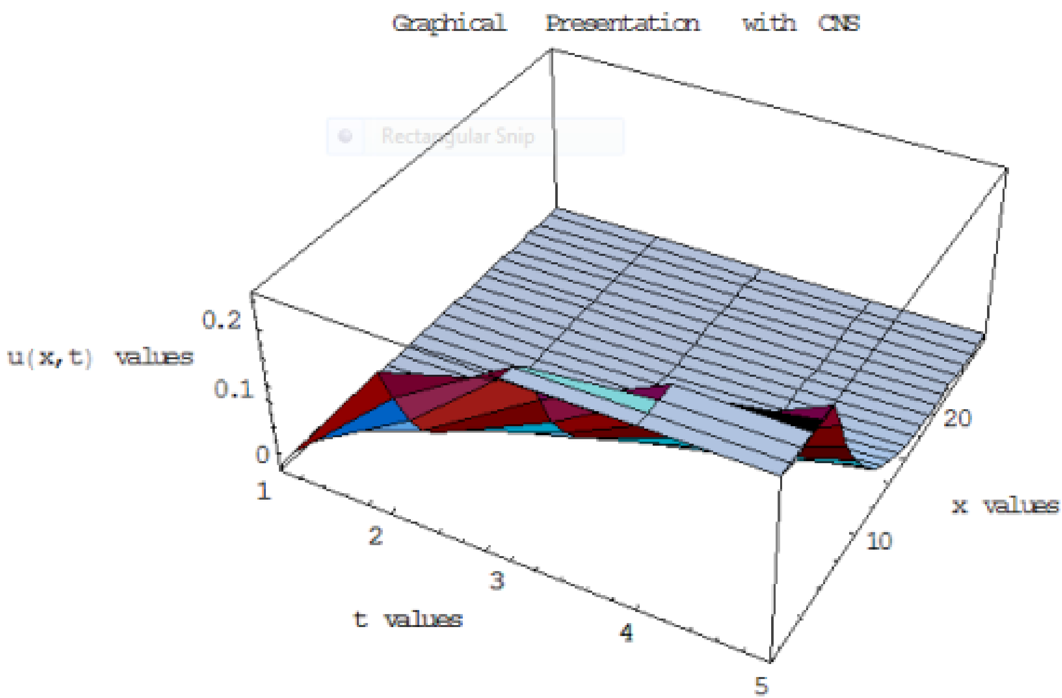

Figure 4: Graphical Presentation Using CNS with β =5 and r=1

8.2 Comparison of Numerical Results

The numerical computational results for solutions of equation (1.1) were obtained. The mesh ratios were varied and their effects on the stability of the two schemes were noted. The results showed that as the mesh ratio reduces, the solutions tend towards the true solution for both schemes, CD-FD and CN-FD. In addition, CD-FD scheme showed consistency with the original equation (1.1). According to fundamental theorem of numerical analysis, a scheme which is stable and shows consistency is convergent, [1].

8.3 Discussion

From presentations of the solutions, it can be observed that,
(a)for a given value of $x, u(x,t) \approx U_{m,n}$ increases to nearly one as t tends to infinity
(b)for a given value of $t, u(x,t) \approx U_{m,n}$ decreases to nearly zero as x tends to infinity. Propagation of waves can be a very slow process. As in ground waves, fluid moves a few meters in several years. That is why the slope of the surface decreases slowly. In this study, we used propagation of waves as seen on guitar strings or a drum head to simulate the waves. Mechanical periodical vibrations always cause sound. The sound waves propagate without losses only

under this assumption of an ideal medium. In reality, sound propagation in a viscous fluid is damped; that is, the amplitude of the pressure of the sound wave decreases with increasing distance from the sound source, [12]. Our results from the two numerical schemes, CD-FD and CN-FD, confirm this since the displacement of the particles given by $u(x, t)$ decrease and tend towards zero with an increase in the distance from the source (in this case at $t = 0$).

8.4 Conclusion

This study focused on the third order viscous sound wave equation. The objectives of this study were: first, to solve the model equation (1.1) using FDM. To pursue this objective, we discretised the equation, developed two numerical schemes, namely CD-FD and CN-FD Schemes. Numerical solution algorithms were developed, and Mathematica software used to generate solutions to the third order viscous wave equation. The study found that both schemes were conditionally stable. We restricted the mesh size ratio ($\beta = k/h^2$ and $r = k^2/h^2$, $(h = \triangle(x), k = \triangle(t))$ and even the upper limit on the values of x. CD-FD and CN-FD analogues were used for u_{xxt} and came up with two sets of results respectively; Case One for a larger mesh ratio and the other for a slightly smaller case two. The two schemes have produced nearly similar results. We managed to come up with the numerical solution system to the viscous wave equation under study and the results interpreted.

8.5 Recommendations

For further research, we recommend the following:
(i) Use of Von Neumann method to test stability analysis of FDM for equation (1.1)
(ii) Explore an analytical solution to this problem using other methods apart from Laplace Transforms.
(iii) Use a centred-central or centred-backwards analogue of u_{xxt} to come up with the schemes and solve the equation then compare the results.
(iv) Implement the finite-difference method on an unstructured grid as well which is rare.
(v) Find a numerical solution to this equation by finite element method and compare the results.

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