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On the Adjacency Matrices of the Anderson-Livingston Zero Divisor Graphs of Galois Rings

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Abstract

A ring is Galois if its subset of all the zero divisors (including zero) forms a principal ideal. Galois rings are generalizations of Galois fields and have been used widely in the past few decades to construct various optimal families of error correcting codes. These rings are important in the structure theory of finite commutative rings. Furthermore, Galois rings are the building blocks of Completely Primary Finite Rings (C.P.F.R) which play a fundamental role in the study of the structures of finite rings in the sense that every finite ring is expressible as a direct sum of matrix rings over Completely Primary Finite Rings. Zero divisor graphs have been of interest to Mathematicians, particularly in understanding the structures of zero divisors in finite rings. However, not much is known about the adjacency matrices of zero divisor graphs of finite rings. In this paper we present some explicit results on the adjacency matrices of the Anderson-Livingston zero divisor graphs of Galois rings.

Mathematics Subject Classification: 05C50, 16P10, 68R10

Keywords: Adjacency matrices, Zero divisor graph, Galois rings

1. Introduction

Studies on the adjacency matrices of zero divisor graphs of finite rings have been done in the recent past. For instance, Pranjal, Amit and Vats in [5] obtained some results on the characterization of adjacency matrix corresponding to the zero divisor graph of finite commutative ring of Gaussian integers modulo n. They calculated the number of zero divisors, examined the nature of the matrix and determined the order of the matrix for specific cases. They obtained that for an adjacency matrix relating to zero divisor graph $\mathbb{Z}_p[i]$, the order of the adjacency matrix in such cases is $2(p-1) \times 2(p-1)$ and the adjacency matrix is always non singular. Sharma, Amit and Vats in [6] investigated the adjacency matrices and the neighborhood associated with the zero divisor graph of finite commutative rings. Some of their findings were that, for $p \neq 2$, the adjacency matrix is always singular for the ring $\mathbb{Z}_p \times \mathbb{Z}_p$. Moreover, the number of the zero divisors in $\mathbb{Z}_p \times \mathbb{Z}_p$ is $2(p-1)$, the eigenvalues of the matrix with respect to the zero divisor graph are $p-1$ and 0. In this paper, we have presented general results on some properties of the adjacency matrices of the Anderson- Livingston zero divisor graphs of Galois ring.

2. Preliminaries

Definition 1[2]. A graph is a structure consisting of a set of vertices, a set of edges and an incident relation, describing the adjacent vertices (vertices joined by edge).
Remark: If the vertices are zero divisors in a ring and an edge exists between x and y when $xy = 0$, then the resultant graph is the Anderson - Livingston zero divisor graph as defined in [1].
Definition 2[2]. Let $G = (V, E)$ be a graph where $V = \{v_1, v_2, \ldots, v_r\}$ is the set of edges, $E = \{e_1, e_2, \ldots, e_r\}$ is the set of edges and there exist no multiplicative edges. The adjacency matrix of G is the $r \times r$ matrix $A = [A_{ij}]$ where

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{elsewhere} \end{cases}$$

for $1 \leq i, j \leq r$.

Remark: Further definitions and notations can be obtained from [2].

3. Adjacency matrix of Anderson - Livingston zero divisor graph of Galois ring

For a Galois ring $R_0 = GR(p^{kr}, p^k)$, the set of non zero divisors is $pR_0 - \{0\}$. Since $|pR_0| = p^{(k-1)r}$, $|pR_0 - 0| = p^{(k-1)r} - 1$.
Therefore, the adjacency matrix of the zero divisor graph of Galois Ring R_0 is of order $(p^{(k-1)r} - 1) \times (p^{(k-1)r} - 1)$.
The results are presented for each characteristic of R_0.

3.1 Adjacency matrix of Anderson - Livingston zero divisor graphs of Galois rings of characteristic p

In this case, $R_0 = GR(p^r, p) \cong \mathbb{F}_{p^r}$ which is a finite field of order p^r for a prime integer p and positive integer r. The ring has no non-zero zero divisors and therefore, the set of vertices $V(\Gamma(R_0)) = \emptyset$ and the adjacency matrix does not exist.

3.2 Adjacency Matrices of Anderson - Livingston Zero divisor graphs of Galois Rings of Characteristic p^2.

Here, $R_0 = GR(p^{2r}, p^2)$. The non-zero zero divisors form the vertex set $V(\Gamma(R_0)) = V(\Gamma(GR(p^{2r}, p^2)))$. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, \overline{\epsilon_2}, \ldots, \overline{\epsilon_r} \in R_0/pR_0$ forms a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then for $\mu = 1, \ldots, r$,
$V(\Gamma(R_0)) = \{\sum_{\mu=1}^{r} a_\mu \epsilon_\mu\} - \{0\}, a_\mu \in \{0, p, \ldots, (p-1)p\}$. It is observed that the number of possible linear combinations in $V(\Gamma(R_0))$ excluding 0, is $p^r - 1$. For the avoidance of zero, the elements of $V(\Gamma(R_0))$ shall be denoted by $u_i, 1 \leq i \leq p^r - 1$.
Since char $R = p^2$, the product of any two zero divisors is zero and therefore, every pair of zero divisors are adjacent. To avoid the existence of loops, the adjacency matrix is represented by $[A_{ij}]$ where

$$A_{ij} = \begin{cases} 1 & u_i \neq u_j \\ 0 & u_i = u_j \end{cases}$$

The order of the matrix is $(p^r - 1) \times (p^r - 1)$. The following result characterizes the determinant of $[A_{ij}]$.

Proposition 1. *Let R_0 be a Galois ring of characteristic p^2 and order p^{2r}. Suppose $[A_{ij}]$ is the adjacency matrix of the zero divisor graph of R_0. Then*

$$det([A_{ij}]) = \begin{cases} p^r - 2 & \textit{if } [A_{ij}] \textit{has even number of columns} \\ 2 - p^r & \textit{if } [A_{ij}] \textit{has odd number of columns} \end{cases}$$

Proof. Clearly,

$$[A_{ij}] = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 0 & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \\ 1 & 1 & \cdot & \cdot & 1 & 0 \end{bmatrix}$$

of order $(p^r - 1) \times (p^r - 1)$. The row equivalent of

$$[A_{ij}] \text{ is } \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ 0 & 0 & 1 & \cdot & \cdot & 1 & 2 \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & & & & 1 & p^r-3 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & p^r-2 \end{bmatrix}$$

If the number of columns of $[A_{ij}]$ is even, then det $([A_{ij}]) = p^r - 2$. For the odd number of columns of $[A_{ij}]$, the determinant is $-(p^r - 2) = 2 - p^r$.

□

Next, we investigate the point spectrum of $[A_{ij}]$.

Proposition 2. *Let R_0 be a Galois ring of characteristic p^2 and order p^{2r}. The point spectrum of the adjacency matrix $[A_{ij}]$ of the zero divisor graph of R_0 is given by*

$$\sigma_{point}([A_{ij}]) = \begin{cases} p^r - 2 & \textit{multiplicity } 1 \\ -1 & \textit{multiplicity } p^r - 2 \end{cases}$$

Proof. The point spectrum of the matrix $[A_{ij}]$ is obtained by computing the eigenvalues λ. Since $[A_{ij}]$ is such that $A_{ij} = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$, the matrix $[\lambda I_{p^r-1} - [A_{ij}]]$ is such that the entry is λ when $i = j$ and -1 when $i \neq j$. The determinant expansion of the matrix $[\lambda I_{p^r-1} - [A_{ij}]]$ yields the characteristic polynomial $\lambda^{p^r-1} - \ldots - (p^r - 2)$ which is monic of degree $p^r - 1$. The characteristic equation $\lambda^{p^r-1} - \ldots - (p^r - 2) = 0$ factorizes into $(\lambda + 1)^{p^r-2}(\lambda - (p^r - 2)) = 0$ whose solution yields the eigenvalues. □

The dimension theorem of the adjacency matrix $[A_{ij}]$

Conventionally, an $n \times m$ matrix is a transformation from the linear space $\mathbb{R}^m$ to the linear space $\mathbb{R}^n$. Since the adjacency matrix of the zero divisor graph of a Galois ring is a square matrix of order $(p^r - 1) \times (p^r - 1)$, then it is a transformation from $\mathbb{R}^{p^r-1}$ to $\mathbb{R}^{p^r-1}$.

Proposition 3. *The row space or column space of the adjacency matrix of the zero divisor graph of a Galois ring of characteristic p^2 is $p^r - 1$ dimensional.*

Proof. The adjacency matrix $[A_{ij}]$ is row equivalent to $\begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 & 0 \\ 0 & 1 & 1 & \cdot & \cdot & 1 & 1 \\ 0 & 0 & 1 & \cdot & \cdot & 1 & 2 \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & & & & 1 & p^r-3 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & p^r-2 \end{bmatrix}$.

The results easily follows from the matrix. □

Definition: The range of $[A_{ij}] = \{b \in \mathbb{R}^{p^r-1} | Ax = b$ for $x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{p^r-1} \end{pmatrix}\}$.

Remarks:
The range of $[A_{ij}] \subseteq$ Column space of $[A_{ij}]$.
Now, let

$$f_i = \begin{cases} 0 & i^{th}\text{position} \\ 1 & \text{elsewhere,} \end{cases}$$

then range of $[A_{ij}] = \text{span}\ \{f_i, 1 \leq i \leq p^r - 1\}$.
Since rank $[A_{ij}] = p^r - 1$, nullity $[A_{ij}] = 0$ and the number columns of $[A_{ij}] = p^r - 1$, the following result summarizes the relationship between the rank $[A_{ij}]$, nullity $[A_{ij}]$ and the columns of $[A_{ij}]$.

Theorem 1. *Let $[A_{ij}]$ be the adjacency matrix of the zero divisor graph of a Galois ring of characteristic p^2 and order p^{2r}. Then rank $[A_{ij}]$+nullity$[A_{ij}]$ = number of columns of $[A_{ij}]$.*

3.3 Adjacency matrix of Anderson - Livingston zero divisor graph of Galois rings of characteristic $p^k, k \geq 3$

In this section, we consider the Galois ring $R_o = GR(p^{kr}, p^k), k \geq 3$. Let $\epsilon_1 \ldots \epsilon_r \in R_0$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, \ldots, \overline{\epsilon_r} \in R_0/pR_0$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield, $\mathbb{F}_p$. The non-zero zero divisors of the vertex set
$V(\Gamma(R_0)) = V(\Gamma(GR(p^{kr}, p^k))) = \{\sum_{\mu=1}^{r} a_\mu \epsilon_\mu, a\mu \in \{0, p, \ldots, (p^{k-1} - 1)p\}$.
The number of possible linear combinations in $V(\Gamma(R_0))$ excluding (0) is $p^{(k-1)r} - 1$. For the avoidance of (0), the elements in $V(\Gamma(R_0))$ shall be denoted by $u_i, 1 \leq i \leq p^{(k-1)r} - 1$. The adjacency matrix is represented by $[A_{ij}]$, where
$A_{ij} = \begin{cases} 1, & u_i \text{ is adjacent } u_j \\ 0, & u_i \text{ is not adjacent } u_j \end{cases}$
The order of the matrix is $(p^{(k-1)r} - 1) \times (p^{(k-1)r} - 1)$. The following result characterizes the determinant of $[A_{ij}]$.

Proposition 4. *Let R_0 be a Galois ring of characteristic p^k and order p^{kr}. Suppose $[A_{ij}]$ is the adjacency matrix of the zero divisor graph of R_0. Then the determinant of $[A_{ij}] = 0$*

Proof. Since the cases do not intersect, we proceed case by case
Case I: $p = 2$

$$\text{The adjacency matrix } [A_{ij}] = \begin{pmatrix} & [T_{lm}] & & & 1 & & [S_{zw}] & \\ & & & & 1 & & & \\ & & & & 1 & & & \\ & & & & \cdot & & & \\ & & & & \cdot & & & \\ 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ & & & & 1 & & & \\ & & & & 1 & & & \\ & [S_{wz}] & & & \cdot & & [T_{ml}] & \\ & & & & \cdot & & & \\ & & & & \cdot & & & \\ & & & & 1 & & & \end{pmatrix}$$

where $1 \leq l, m < 2^{(k-2)r}; 2^{(k-2)r} < w, z < 2^{(k-1)r}$.

$$\text{Now } T_{lm} = \begin{cases} 1 & u_i = 4\epsilon_\mu, u_m = 4\epsilon_\tau, 1 \leq \mu, \tau < 2^{(k-2)r}, u_l \neq u_m \\ 0 & u_l = u_m; u_l \text{ is not adjacent to } u_m \end{cases}$$

$$\text{and } S_{wz} = \begin{cases} 1 & u_w = 4\epsilon_\mu, u_z = 4\epsilon_\tau, 1 \leq \mu, \tau < 2^{(k-2)r}, u_w \neq u_z \\ 0 & u_w = u_z; u_w \text{ is not adjacent to } u_z \end{cases}$$

Without loss of generality, assume $u_i \neq 4\epsilon_\mu$, for $1 \leq \mu \leq r$, then the only vertex adjacent to u_i is 2^{k-1}. So any vertex which is a multiple of 2 but not a multiple of 4 is only adjacent to 2^{k-1}. The cardinality of such vertices is $2^{(k-2)r}$.
Now, the row equivalent matrix of $[A_{ij}]$ has $(111\ldots101\ldots1)$ as its first row, followed by non-zero rows and $2^{(k-2)r} - 1$ zero rows at the bottom. Expansion along any zero row yields a determinant of zero.

Case II: *p is odd.*
The adjacency matrix $[A_{ij}]$ consists of the following categories of rows:
(i) Rows formed by the product of the vertices which are linear combinations of p over R_0/pR_o. These vertices are only adjacent to linear combinations of $p^{(k-1)}$.
(ii) Rows formed by the products of the vertices which are linear combinations of p^2 over R_0/pR_0.
(iii) Rows formed by the products of vertices which are linear combinations of p^{k-1} over R_0/pR_0.
Now, the row equivalent matrix of $[A_{ij}]$ has $(0\ldots010\ldots010\ldots0\ldots10\ldots0)$ in the first row, followed by non-zero rows and $p^{(k-1)r} - p^{(k-2)r}$ zero rows at the bottom. Expansion along a zero row yields a determinant of zero. □

In the next result, we investigate the point spectrum of $[A_{ij}]$.

Proposition 5. *Let R_o be a Galois ring of characteristic p^k and order $p^{kr}, k \geq 3$ Then $0 \in \sigma_{point}([A_{ij}])$.*

Proof. Suppose $[A_{ij}]$ is the adjacency matrix of the zero divisor graph of R_0.

Then, the matrix $[\lambda I_{p^{(k-1)r}-1} - [A_{ij}]] =$
$$\begin{pmatrix}
\lambda & 0 & \cdots & -1 & 0 & \cdots & -1 & 0 & \cdots & 0 \\
0 & & & & & & & & & . \\
. & & & & & & & & & . \\
. & & & & & & & & & . \\
-1 & & & & & & & & & -1 \\
. & & & & \lambda & & & & & . \\
0 & & & & & & & & & . \\
-1 & & & & & & & & & . \\
. & & & & & & & & & 0 \\
0 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & 0 & \lambda
\end{pmatrix}.$$

The determinant expansion of $[\lambda I_{p^{(k-1)r}-1} - [A_{ij}]]$ yields

$$\lambda^{p^{(k-1)r}-1} + a_{p^{(k-1)r}-2}\lambda^{p^{(k-1)r}-2} + \ldots + a_1\lambda$$

$$= \lambda(\lambda^{p^{(k-1)r}-2} + a_{p^{(k-1)r}-2}\lambda^{p^{(k-1)r}-3} + \ldots + a_1)$$

Equating the polynomial to zero yields $\lambda = 0$ as one of the eigenvalues.

$\square$

Remark: The adjacency matrices of the graphs considered in this section are transformations from $\mathbb{R}^{p^{(k-1)r}-1}$ to $\mathbb{R}^{p^{(k-1)r}-1}$.
The row space and the column space of $[A_{ij}]$ are subspaces of $\mathbb{R}^{p^{(k-1)r}-1}$

Proposition 6. *The row space of the adjacency matrix of the zero divisor graph of the Galois ring of order p^{kr} and characteristic $p^k, k \geq 3$ is $p^{(k-2)r} - 1$ dimensional.*

Proof. In the row equivalent matrix of $[A_{ij}]$, there are $p^{(k-1)r} - p^{(k-2)r}$ zero rows at the bottom. So, the number of elements in the basis of the row space of $[A_{ij}]$ = number of columns in the row equivalent matrix - number of zero row $= p^{(k-1)r} - 1 - (p^{(k-1)r} - p^{(k-2)r}) = p^{(k-2)r} - 1$

$\square$

Proposition 7. *For $k \geq 3$, the nullity of the adjacency matrix of the zero divisor graph of a Galois ring of characteristic p^k and order p^{kr} is $p^{(k-1)r} - p^{(k-2)r}$*

Proof. The result follows from the dimension theorem, since rank $([A_{ij}]) = p^{(k-2)r} - 1$ $\square$

Conclusion

This study has exposed the following properties of adjacency matrices of zero divisor graphs of Galois rings: determinant, point spectrum, rank and nullity. Further research may yield results on the properties of adjacency matrices of the zero divisor graphs of Galois ring extensions.

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